The Mathematics of Framed Bridge Structures

Scenario

You are a graduate structural engineer working for a major consultancy company that is an acknowledged world-leader in its field. The company employs 6000 staff and operates from 70 offices in 40 countries throughout the world and has an annual turnover of some £500 million per year.

You are working within a team of specialist bridge design engineers who are responsible for a large portfolio of bridges being designed for construction in the UK and in a number of overseas developing countries. You have been given the task of designing a steel bridge to carry a railway across a major motorway in the South of England. The form of the bridge will be a triangulated pin-jointed steel truss (see photo 4 below) – this has been determined to be the most appropriate form of bridge chosen from the many different types of bridge structure used throughout the world.

As part of the initial design process you are required to analyse the proposed bridge and determine the forces within each of the bridge members. The calculation of these forces will subsequently lead to the selection of the most appropriate size of steel section for the members and the detailed design of the connections where the members meet.

In this exercise the learner will apply techniques for the solution of simultaneous equations using methods, including matrix techniques, that form the basis of the calculations that would be undertaken by the structural engineer, often now a days carried out using powerful computer software packages.

Importance of Exemplar in Real Life

Bridges range from small structures such as simple footbridges to iconic structures such as the Humber Bridge which, when opened in 1981, held a 17 year world record for being the longest single span suspension bridge in the world. Built at a cost in excess of £150m its world record and cost of construction have since been far exceeded – the record for the longest span suspension bridge currently being the Akashi-Kaikyō Bridge in Japan which has a suspended centre span of nearly 2000 metres, although such records are being continuously broken.

Although the construction of such iconic bridges is often a statement of national pride the decision to construct any bridge is usually based on social and economic criteria. For example, the construction of the Humber Bridge provided access to two areas of the UK which were geographically remote and provided the opportunity for commercial, industrial and tourist development and saved many millions of vehicle miles in providing a short transit route between both sides of the Humber estuary. Such potential developments and financial savings are the justification for the huge cost investments made in building bridges throughout the world although factored into such cost considerations has to be the ongoing cost of maintenance and repair over the life span of the bridge.

Bridges are constructed in a wide range of materials including masonry, timber, steel, reinforced concrete, prestressed concrete and composite construction. They are built in a variety of different forms including simple beams, trusses, arches, suspension, and cable-stayed structures. The choice of material and structural form depends on a wide range of factors such as the load to be carried including the weight of vehicles, the span length of the bridge, the construction and maintenance costs, the visual impact and so on. Whatever the final choice the Structural Engineer makes, either individually or as part of a design team has an important role to play in ensuring the most appropriate choice of structure and materials and in ensuring the structural integrity of the bridge during construction and throughout its working life.

Examples of different types of bridges can be seen in figures 1 to 4. Figure 4 illustrates the type of bridge that we will be considering in this exercise. It is fabricated from structural steel sections to form a bridge which is *simply supported* i.e resting and supported on a supporting structure at either end.

Figure 1: *Suspension Bridge*

Figure 2: *Cable Stayed Bridge*

Figure 3: *Reinforced Concrete Bridge with masonry arch bridge in background*

Figure 4: *Steel Truss Road Bridge*

Figure 5: *Space Frame Roof Structure*

It should be noted that this type of triangulated structure is used not only in bridge construction but also in many other forms of construction such as the space-frame roof structure shown in figures 5 above.

Background Theory

In analysing the type of bridge structure shown in figure 4 above a number of simplifying assumptions are made including:

- the three dimensional bridge structure is idealised as a two dimensional structural model as shown in figure 6 below;
- the structure is formed from a series of braced triangular frames;
- all loads (weight of traffic etc) are transferred to the structure through the *nodes* or *joints* i.e where the members meet and
- the only forces in the members are either axial tension forces (putting a member into a state of tension) or axial compression forces (putting a member into a state of compression)

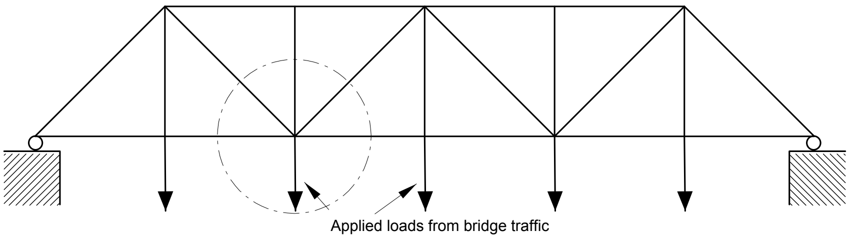

Figure 6(a): Two dimensional idealised model of the bridge shown in figure 4

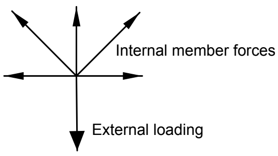

Figure 6(b): Single joint taken in isolation showing both internal and external forces

The approach to analysing the structure is to consider the *equilibrium* of each *joint* of the structure as show in figure 6(b) above. The general engineering principle to be applied is that at each joint equilibrium is assured if the components of all forces acting in (a) the horizontal direction and (b) the vertical direction summate to zero. In mathematical terms this can be expressed as:

$$\sum H = 0 \qquad (1)$$

and $$\sum V = 0 \qquad (2)$$

The application of these expressions will give two equations which can be solved for two unknown forces. Hence, if at any joint there are no more than two unknown member forces then there will be two simultaneous equations that can be solved for the two unknowns.

For example, figure 7(a) below shows a joint connecting two members inclined at 60° and 45° respectively with an external applied force of 20kN acting in a downwards direction and 10kN acting to the right. The unknown forces in the two members are shown as F_1 and F_2 and are shown as acting away from the joint which is a convention that assumes that the members are acting in *tension*. If subsequent calculation indicates that the value of force in either of these members is negative then this indicates that the members are in compression. Note that, although a rigorous sign convention can be established for both clockwise and anti-clockwise angles it is best to visualise the problem by sketching the force component diagrams as shown in figure 7(b).

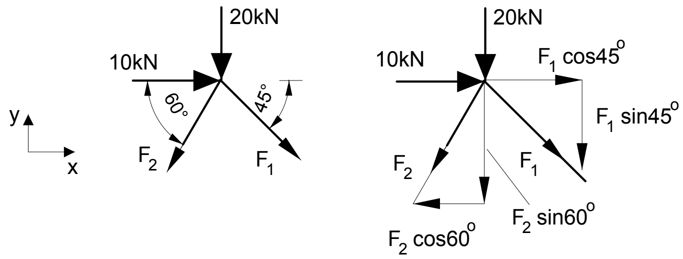

Figure 7: (a) Forces on a typical single joint (b) Forces resolved into vertical and horizontal components

Figure 7(b) shows the two unknown forces resolved into their horizontal and vertical *components* of force and the application of equations (1) and (2) will give:

$$\sum H = 0: \qquad F_1 Cos45^o - F_2 Cos60^o + 10 = 0 \qquad (3)$$

$$\sum V = 0: \qquad -F_1 Sin45^o - F_2 Sin60^o - 20 = 0 \qquad (4)$$

Note that the sign convention assumes that when resolving forces vertically all *upward* forces are *positive*. Likewise when resolving horizontally all forces to the *right* are positive. Equations (3) and (4) can be written in matrix form as:

$$\begin{vmatrix} Cos45^o & -Cos60^o \\ -Sin45^o & -Sin60^0 \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \end{vmatrix} = \begin{bmatrix} -10 \\ 20 \end{bmatrix} \qquad (5)$$

The above represents a pair of simultaneous equations expressed in matrix format and can be solved for the two unknowns, F_1 and F_2, using standard matrix inversion techniques.

The above approach lends itself to the systematic solution of all the forces in a typical triangulated frame providing that at any joint there are no more than two unknown forces. In practice this means starting at a joint where there are only two unknown forces and, having calculated the two unknown forces at that joint, transfer the values of the calculated forces to the joints at the far end of each member, thus reducing the number of unknowns at the far end by one. Once a second joint has been identified as having only two unknown forces then the procedure can be repeated until all member forces are known.

In the case of more complex triangulated structures it is possible that it can not be analysed in this way because there are too many members and hence too many unknown forces for sufficient independent equations to be written down. In such a case the structure is referred to as *statically indeterminate* and more sophisticated analytical techniques must be used.

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Questions

Example data (1) : For the joint shown in figure 8 calculate the forces F_1 and F_2 for the given sets of data

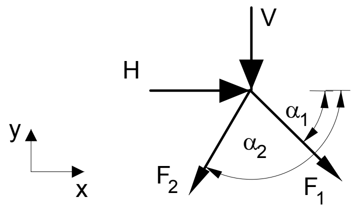

Figure 8

α_1	α_2	**H**	**V**	**F1**	**F2**
degrees	degrees	kN	kN	kN	kN
30	120	50	-100		
30	120	100	-100		
45	90	200	-100		
30	125	-50	-150		
60	135	200	-200		

Example data (2) : For the crane structure shown in figure 9 calculate all the member forces. Inclined members are at 45°. For this problem solve joint A and use the solution of F_2 from joint A to calculate the forces at joint B. This can be done as two separate sets of matrix solutions or a large 4x4 matrix can be set up and solved in terms of the four unknowns F_1 to F_4. The problem can be repeated for other member inclinations.

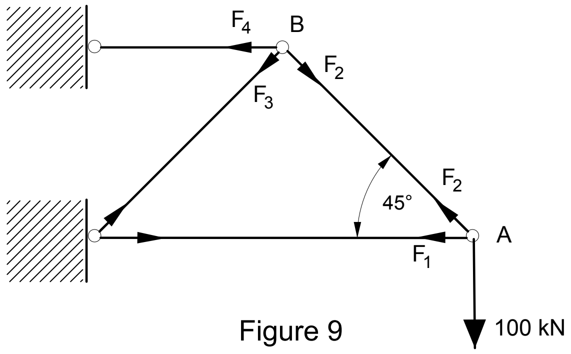

Figure 9

Where to find more

1. Ray Hulse & Jack Cain, *Structural Mechanics,* 2nd edn, Palgrave, 2000. (ISBN 0-333-80457-0)
2. John Bird, *Engineering Mathematics,* 5th edn, John Bird, 2007 (ISBN 978-07506-8555-9)

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The Royal Academy of Engineering

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CONSTRUCTION AND THE BUILT ENVIRONMENT DIPLOMA

INFORMATION FOR TEACHERS

Teachers will need to understand and explain the theory outlined above and have knowledge of:

- Some terminology relating to structural design and construction
- The concept of vector components of force
- Solution of simultaneous equations using matrix techniques

Topics covered from Mathematics for Engineers

- Topic 1: Mathematical Models in Engineering
- Topic 7: Linear Algebra and Algebraic Processes

Learning Outcomes

- LO 01: Understand the idea of mathematical modelling
- LO 07: Understand the methods of linear algebra and know how to use algebraic processes
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

Assessment Criteria

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 7.1: Solve engineering problems using vector methods
- AC 7.2: Use matrices to solve two simultaneous equations in two unknowns
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

Links to other units of the Advanced Diploma in Construction & The Built Environment

Unit 3	Civil Engineering Construction
Unit 29	Science and materials in construction and the Built Environment
Unit 30	Structural Mechanics
Unit 31	Design

Solution to the Questions

Example data (1) :

α_1	α_2	**H**	**V**	**F1**	**F2**
degrees	degrees	kN	kN	kN	kN
30	120	50	-100	-93.30	-61.60
30	120	100	-100	-136.60	-36.60
45	90	200	-100	-282.84	100.00
30	125	-50	-150	-45.25	-155.50
60	135	200	-200	-292.82	75.79

Example data (2) : F_1 = -100kN, F_2 = 141.42kN, F_3= - 141.42kN, F_4 = 200kN

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