The Mathematics of Pumping Water

AECOM Design Build

Civil, Mechanical Engineering

Please observe the conversion of units in calculations throughout this exemplar.

INTRODUCTION

In any pumping system, the role of the pump is to provide sufficient pressure to overcome the operating pressure of the system to move fluid at a required flow rate. The operating pressure of the system is a function of the flow through the system and the arrangement of the system in terms of the pipe length, fittings, pipe size, the change in liquid elevation, pressure on the liquid surface, etc. To achieve a required flow through a pumping system, we need to calculate what the operating pressure of the system will be to select a suitable pump.

Figure 1: Typical Vertical Turbine Water Pumps

MATHEMATICAL MODEL AND CALCULATIONS

Consider the pumping arrangement shown in Figure 2 below:

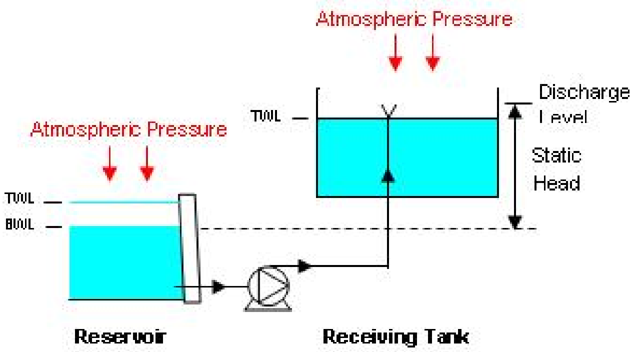


Figure 2: Pumping Arrangement

Water is pumped from the reservoir into a receiving tank. This kind of arrangement is used to lift water from a reservoir, or river, into a water treatment works for treatment before the water goes into the supply network. The water level in the reservoir varies but the discharge level in the receiving tanks remains constant as the water is discharged from a point above the water level. The pump is required to pass forward a flow of 2500 m^3/hr to the receiving tank.

The operating pressure of a pumped system is calculated in the SI unit of meters (m). To maintain dimensional consistency, any pressure values used within the calculations are therefore converted from kPa into m using the following conversion;

1 kPa = 0.102 m

(as measured by a water filed U tube manometer)

For the above system, the operating pressure or the total system head, H_{Total} , is defined as:

$$H_{Total} = H_s + H_D + (P_{RT} - P_{RES}) \quad \dots (1)$$

where,

H_s = Static head (m)

H_D = Dynamic head (m)

P_{RT} = Pressure on the surface of the water in the receiving tank (m)

P_{RES} = Pressure on the surface of the water in the reservoir (m)

Although the atmospheric pressure changes with height, the change in pressure that occurs over the pumping height is often so small that it can be considered negligible. In this exemplar, the change in pressure over the elevation from the reservoir to the receiving tank is not that significant and hence is negligible, i.e., $P_{RT} - P_{RES} \approx 0$.

Therefore, equation (1) becomes:

$$H_{Total} = H_s + H_D \quad \dots (2)$$

The static head H_s is the physical change in elevation between the surface of the reservoir and the point of discharge into the receiving tank. As the water level in the reservoir can vary, the static head for the system will vary between a maximum and a minimum value:

$$H_{S_{min}} = \text{discharge level} - \text{reservoir TWL}$$

and

$$H_{S_{max}} = \text{discharge level} - \text{reservoir BWL}$$

where

TWL = Top Water Level (reservoir)

BWL = Bottom Water Level (reservoir)

If the discharge point is at a level of 110.5 m above the mean sea level (also known as ***Above Ordnance Datum (AOD)*** in technical language) and the reservoir level varies between 105.2 m AOD and 101.6 m AOD, then:

$$H_{S_{min}} = 110.5 - 105.2 = 5.3 \text{ m}$$

$$H_{S_{max}} = 110.5 - 101.6 = 8.9 \text{ m}$$

As a result of the variation in the static head, the total system head, H_{Total}, will also have a maximum and minimum value which we need to calculate here.

The dynamic head is generated as a result of friction within the system. The dynamic head is calculated using the basic Darcy Weisbach equation given by:

$$H_D = \frac{Kv^2}{2g} \quad \dots (3)$$

where

K = loss coefficient

v = velocity in the pipe (m/sec)

g = acceleration due to gravity (m/sec^2)

We can calculate the velocity in pipe using the following formula:

$$v = \frac{Q}{A} \quad \dots (4)$$

where

Q = flow rate through the pipe (m^3/sec)

A = pipe cross sectional area (CSA) (m^2)

If Q is 2500 m^3/hr and the flow is pumped through a 0.8 m diameter pipe then:

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 0.8^2}{4} = 0.5 \text{ m}^2$$

Hence, using equation (4), we get:

$$v = \frac{25000}{3600} \times \frac{1}{0.5} = 1.39 \text{ m/sec}$$

The loss coefficient K is made up of two elements:

$$K = K_{\text{fittings}} + K_{\text{pipe}} \quad \dots (5)$$

K_{fittings} is associated with the fittings used in the pipeworks of the system to pump the water from reservoir to the receiving tank. Values can be obtained from standard tables and a total K_{fittings} value can be calculated by adding all the K_{fittings} values for each individual fitting within the system. The following table shows the calculation of K_{fittings} for the system under consideration:

Fitting Items	**No. of Items**	K_{fittings} **Value**	**Item Total**
Pipe Entrance (bellmouth)	1	0.05	0.05
90° Bend (short radius)	10	0.75	7.5
45° Bend (short radius)	2	0.3	0.6
Butterfly Valve (Fully Open)	2	0.3	0.6
Non Return Valve	1	1.00	1.00
Bellmouth Outlet	1	0.2	0.2
Total K_{fittings} **Value**			**9.95**

Table 1: Calculating K_{fittings} for the system under consideration

Hence, the total K_{fittings} for the system under consideration is 9.95.

K_{pipe} is associated with the straight lengths of pipe used within the system and is defined as:

$$K_{\text{pipe}} = \frac{fL}{D} \quad \dots (6)$$

where

f = friction coefficient

L = pipe length (m)

D = pipe diameter (m)

The friction coefficient f can be found using a modified version of the Colebrook White equation:

$$f = \frac{0.25}{\left[\log\left\{\frac{k}{3.7 \times D} + \frac{5.74}{\text{Re}^{0.9}}\right\}\right]^2} \quad \dots (7)$$

where

k = Roughness factor (m)

Re = Reynolds number

The pipe roughness factor k is a standard value obtained from standard tables and is based upon the material of the pipe, including any internal coatings, and the internal condition of the pipeline i.e. good, normal or poor.

Reynolds number is a dimensionless quantity associated with the smoothness of flow of a fluid and relating to the energy absorbed within the fluid as it moves. For any flow in pipe, Reynolds number can be calculated using the following formula:

$$\mathrm{Re} = \frac{vD}{\upsilon} \quad \dots (8)$$

where

υ = Kinematic viscosity (m^2/s)

If the total pipe length is 250 m, the pipe has a roughness factor of 0.3 mm and the kinematic viscosity of water is 1.31×10^{-6} m^2/sec, then from equation (8), we get:

$$\mathrm{Re} = \frac{1.39 \times 0.8}{1.31 \times 10^{-6}} = 8.49 \times 10^5$$

Using this value in equation (7), we get:

$$f = \frac{0.25}{\left[\log\left\{\frac{0.0003}{3.7 \times 0.8} + \frac{5.74}{\left(8.49 \times 10^5\right)^{0.9}}\right\}\right]^2}$$
$$= 0.0165$$

Using this value in equation (6), we get:

$$K_{\text{pipe}} = \frac{0.0165 \times 250}{0.8} = 5.16$$

Finally, using equation (5), the total K value for the system is:

$$K = 5.16 + 9.95 = 15.11$$

We can now calculate the dynamic head using equation (3) as follows:

$$H_D = \frac{15.11 \times (1.39)^2}{2 \times 9.81} = 1.49\,\text{m}$$

The dynamic head is the same for both the maximum and minimum static head conditions as the dynamic head is independent of the system elevation.

Hence, the maximum and minimum total head values for the system at a flow of 2500 m^3/hr can now be calculated using equation (2):

$$H_{Total_{\max}} = 8.9 + 1.49 = 10.39\,\text{m}$$

$$H_{Total_{\min}} = 5.3 + 1.49 = 6.79\,\text{m}$$

Hence we can conclude that in order to pump 2500 m^3/hr at the bottom level in the reservoir, the pump will need to overcome a system pressure of 10.39 m. At the top level, the pump will only need to overcome a system pressure of 6.79 m. If a centrifugal pump were selected to achieve either the maximum or minimum head condition, this would likely result in either too much or too little flow at the other head condition. Instead, if we use a variable speed pump by adjusting the pump speed we can control the flow to the receiving tank to 2500 m^3/hr over the entire head range.

PUMP SELECTION

By repeating the calculation for H_D for a range of flows we can generate a pair of system curves that define the relationship between head and flow for the top and bottom water conditions. These curves define the envelope of the pumping system.

A pump has been selected from manufacturer's details that can achieve the required flow at the BWL at a speed of 675 rpm. The characteristic hydraulic curve for the selected pump has been overlaid onto the system curves (see Figure-3 on the next page) and the effect of running the pump at this speed but at the TWL can be seen. The Intersection of the TWL and BWL System Curves with the Speed Curves define the Pump's maximum and minimum operating speeds. In this instance, the pump would run to the right hand end of its hydraulic curve possibly causing cavitations.

The pump speed needs to be reduced in order to achieve the required flow at the TWL and the required speed can be calculated using the affinity laws:

First affinity law – Flow is proportional to the shaft speed, i.e.,

$$\frac{Q_1}{Q_2} = \frac{N_1}{N_2} \quad \dots (9)$$

where

Q = Flow through the pipe (m^3/sec)

N = Shaft speed (rpm)

Second affinity law – Head is proportional to the square of the shaft speed, i.e.,

$$\frac{H_1}{H_2} = \frac{(N_1)^2}{(N_2)^2} \quad \dots (10)$$

where

H = Head (m)

Using an iterative process of adjusting the pump speed and calculating the resultant flow and head using the above laws, we can determine the required speed of the pump for the TWL condition. In this case, the pump needs to run at around 590 rpm.

The power requirement for the pump can be calculated by:

$$P = \frac{Q \times H \times g \times \rho}{Pump\,Efficiency} \quad \dots (11)$$

where

P = Power (W)

ρ = Density (Kg/m^3)

= 1000 kg/m^3 for water

For this pump, at the maximum head of 10.39 m and a flow of 2500 m^3/hr (0.694m^3/s) the pump efficiency is 84%. Therefore, using equation (11), the power requirement is:

$$P = \frac{0.694 \times 10.39 \times 9.81 \times 1000}{0.84} \text{, or}$$

$$P = 84210\,\text{W} = 84.21k\text{W}$$

Hence, we can say that to overcome the required head of 10.39 m, we need a variable speed pump with 84.21 W.

CONCLUSION

The accurate calculation of the maximum and minimum total head is critical for the selection of a suitable pump. Selection of an unsuitable pump can result in too much or too little water being pumped. Too little water might, for example, result in customers not receiving clean drinking water when they turn on the tap. Too much water might result in water being wasted or even lead to flooding.

The operating pressure of a pumping system can vary due to various factors, e.g. changes in reservoir level, so all the relevant operating conditions need to be assessed to ensure the selected pump is capable of achieving the entire operating range. Using variable speed pumps is one way of coping with the variations in system operating pressure.

EXTENSION ACTIVITIES

1. Calculate $H_{Total_{max}}$ and $H_{Total_{min}}$ for the system if the flow is reduced to 2000m^3/hr.
2. What happens to the pump power if the pump efficiency reduces?
3. Calculate the power requirement of the pump for the following efficiencies:
   - 95%
   - 75%
   - 50%

WHERE TO FIND MORE

1. *Basic Engineering Mathematics*, John Bird, 2007, published by Elsevier Ltd.
2. *Engineering Mathematics*, Fifth Edition, John Bird, 2007, published by Elsevier Ltd.
3. *Pressure and Head Losses in Pipes and Ducts*, D.S. Miller, 1984

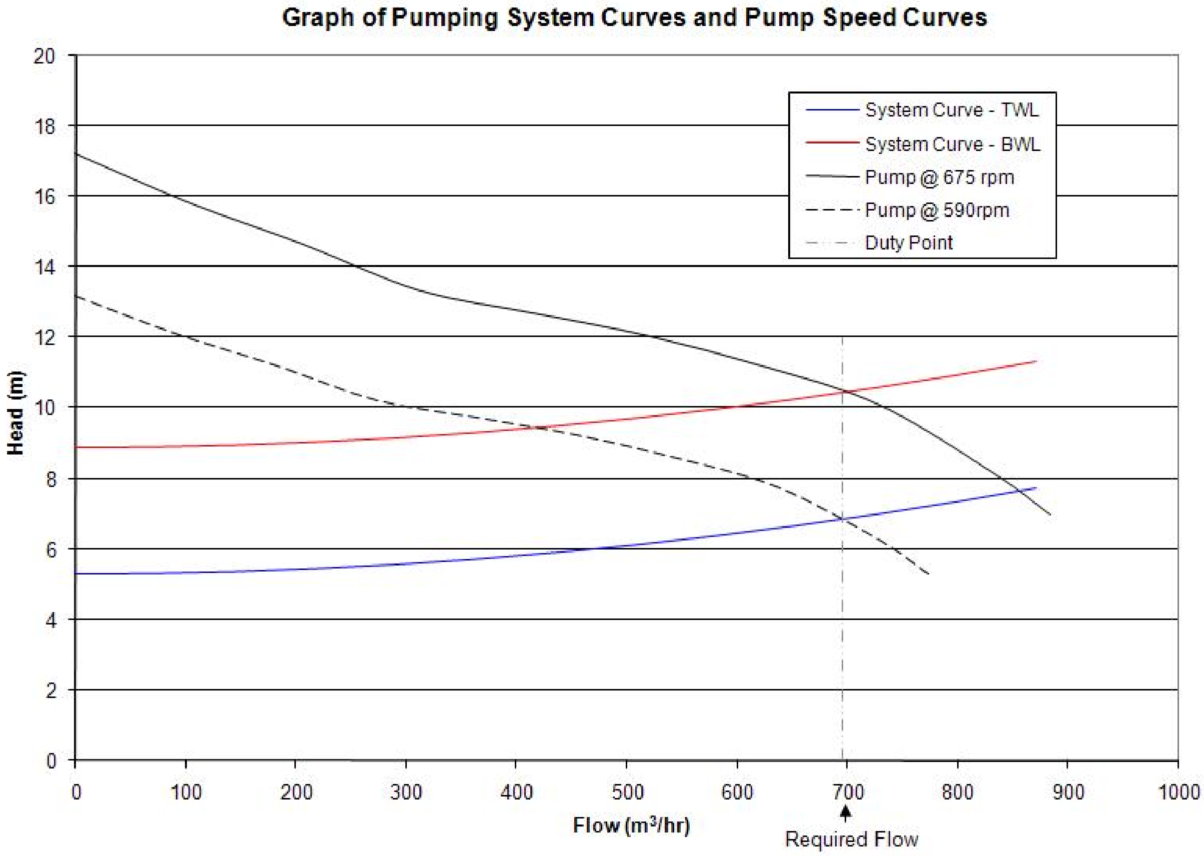


Figure 3: Graph of Pumping System Pressure Curves and Pump Operating Speed Curves

Mathew Milnes – Project Engineer, AECOM Design Build

Mathew has worked in the Water Industry designing clean and dirty water treatment plants for the last 10 years. As a Chartered Mechanical Engineer he uses mathematics on a daily basis to calculate the size and performance of process equipment to provide people with clean drinking water and to ensure their wastewater is treated and disposed of in an environmentally acceptable way.

INFORMATION FOR TEACHERS

The teachers should have some knowledge of

- terminology used in pumping water and the physical meaning behind them
- handling formulae with the method of back-substitution
- plotting graphs using excel sheets
- manipulating calculations and converting units for uniformity
- Equation 1 can be derived from an extension to the Euler equation. Please refer to the last page for more information and observe the use of partial derivatives and limit theory.

TOPICS COVERED FROM "MATHEMATICS FOR ENGINEERING"

- Topic 1: Mathematical Models in Engineering
- Topic 4: Functions

LEARNING OUTCOMES

- LO 01: Understand the idea of mathematical modelling
- LO 04: Understand the mathematical structure of a range of functions and be familiar with their graphs
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

ASSESSMENT CRITERIA

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 4.1: Identify and describe functions and their graphs
- AC 4.2: Analyse functions represented by polynomial equations
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

LINKS TO OTHER UNITS OF THE ADVANCED DIPLOMA IN ENGINEERING

- Unit-1: Investigating Engineering Business and the Environment
- Unit-3: Selection and Application of Engineering Materials
- Unit-4: Instrumentation and Control Engineering
- Unit-5: Maintaining Engineering Plant, Equipment and Systems
- Unit-6: Investigating Modern Manufacturing Techniques used in Engineering
- Unit-7: Innovative Design and Enterprise
- Unit-8: Mathematical Techniques and Applications for Engineers
- Unit-9: Principles and Application of Engineering Science

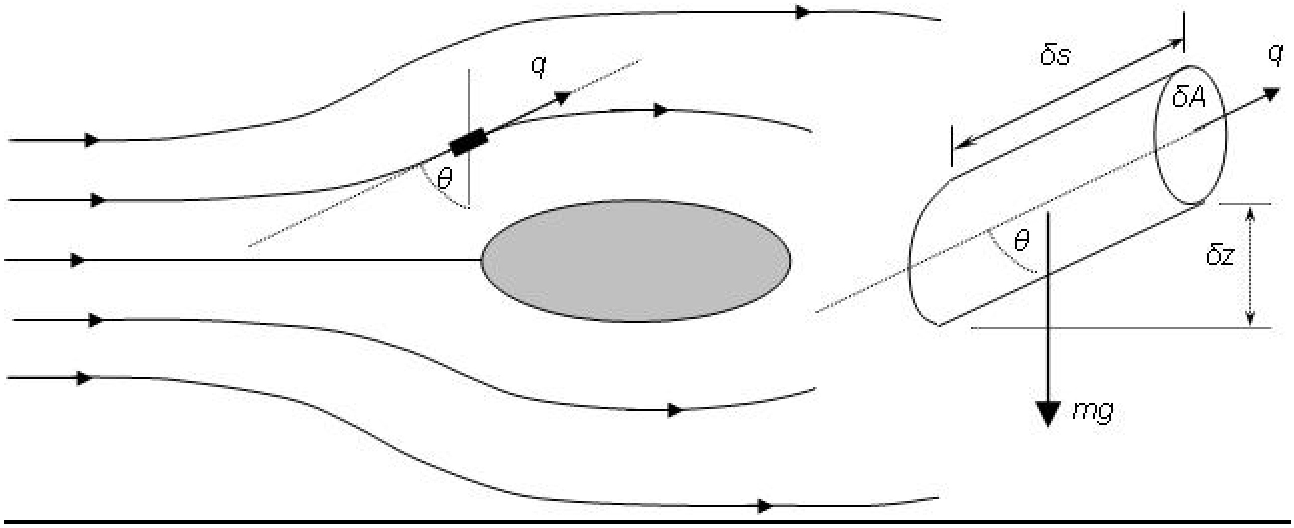
ANSWERS TO EXTENSION ACTIVITIES

1. $H_{Total_{max}}$ = 9.85m, $H_{Total_{min}}$ = 6.25m (H_D = 0.95m)
2. The power requirement goes up as the efficiency reduces.
3. 90% = 74.5 kW, 75% = 94.3kW, 50% = 141.5kW

ANNEXE: EXTENSION OF EULER EQUATION

In this section, we investigate incompressible flow along a streamline under the action of pressure gradients and gravitational body forces - but **not** friction. Hence density is constant and there are no shear forces. During the derivation it will also be necessary to assume that the flow is **steady**.

Consider a small cylindrical element of fluid aligned **along a streamline**. It has a cross sectional area ***dA***, pressure is assumed uniform across its ends ***dA***, and the local velocity is defined ***q***.



Applying Newton's laws of motion to the flow through the cylindrical element along the streamline, the force (in the direction of motion along the streamline) = mass x acceleration.

The mass of the element and the forces acting on it will be considered later, but first we look at the acceleration of the fluid element. Ignoring the possibility that the flow might be steady, ***q*** can change with time ***t***, and also with position ***s*** along the streamline. In other words, ***q*** is a function of ***t* and *s***, or ***q* = f (*t*, *s*)**. Hence, if the element moves a distance δ***s*** in time δ***t***, then the total change in velocity δ***q*** is given by:

$$\delta q = \frac{\partial q}{\partial s}\,\delta s \;+\; \frac{\partial q}{\partial t}\,\delta t$$

and in the limit as δ***t*** tends to zero, the "substantive" derivative is given as:

$$\frac{dq}{dt} = Lim_{\delta t \to 0}\frac{\delta q}{\delta t} = \frac{\partial q}{\partial s} Lim_{\delta t \to 0}\frac{\delta s}{\delta t} + \frac{\partial q}{\partial t} = q\,\frac{\partial q}{\partial s} + \frac{\partial q}{\partial t}$$

In other words, fluid can accelerate because it is moving (at velocity ***q***) through a region with changing velocity, or because the flow is changing with time. However, for a steady flow the local velocity at a point does not vary with time, so the last term under such circumstances will be zero.

Looking now at the forces acting on the element and applying Newton's laws:

$$p\delta A - (p + \frac{\partial p}{\partial s}\delta s)\delta A - \rho\,\delta A\,\delta s\,g\,\cos\theta = \rho\,\delta s\,\delta A\,q\,\frac{dq}{ds}$$

dividing through by δ***A.*** δ***s*** and defining **δ*z* = δ*s* cosθ** , we have that:

$$\frac{\partial p}{\partial s} + \rho\,q\,\frac{\partial q}{\partial s} + \rho\,g\,\frac{\delta z}{\delta s} = 0$$

dividing through by δ***A.*** δ***s*** and defining **δ*z* = δ*s* cosθ** , we have that:

$$\frac{\partial p}{\partial s} + \rho\,q\,\frac{\partial q}{\partial s} + \rho\,g\,\frac{\delta z}{\delta s} = 0$$

and in the limit as δ***s* tends to zero,**

$$\frac{dp}{ds} + \rho\,q\,\frac{dq}{ds} + \rho\,g\,\frac{dz}{ds} = 0$$

$$or \quad \frac{1}{\rho}\frac{dp}{ds} + q\frac{dq}{ds} + g\frac{dz}{ds} = 0$$

This is a form of Euler's equation, and relates *p*, *q*, and *z* along a streamline.

Assuming ρ is constant, and remembering that:

$$q\frac{dq}{ds} = \frac{1}{2}\frac{d(q^2)}{ds}$$

if the term above is substituted into Euler's equation, it then becomes possible to integrate it - giving:

$$\frac{p}{\rho} + \frac{1}{2}q^2 + g\,z = constant\ along\ a\ streamline$$

$$p + \frac{1}{2}\rho\,q^2 + \rho\,g\,z = constant\ a\ streamline$$

$$\frac{p}{\rho\,g} + \frac{q^2}{2\,g} + z = constant\ along\ a\ streamline$$

The three equations above are valid for incompressible, frictionless steady flow, and what they state is that total energy is conserved along a streamline.

The first of these forms of the **Bernoulli equation** is a measure of energy per unit mass, the second of energy per unit volume, and the third of "head", equivalent to energy per unit weight.

In the second equation, the term ***p*** is the static pressure, **{**$\frac{1}{2}\rho q^2$**}** is the dynamic pressure, ρgz is the elevational term, and the **SUM** of all three is known as the **stagnation (or total) pressure,** p_0

In the third equation, $p/\rho g$ is known as the pressure head, $q^2/2g$ as the dynamic head, and the sum of the three terms as the **Total Head *H***.

The Bernoulli equation is used widely in fluid mechanics and hydraulics. Assuming that the flow is actually frictionless and incompressible, what it shows is that if the **velocity falls** in a flow, then the **pressure must rise** - and vice versa.

For a gas, the elevational terms can be assumed negligible.

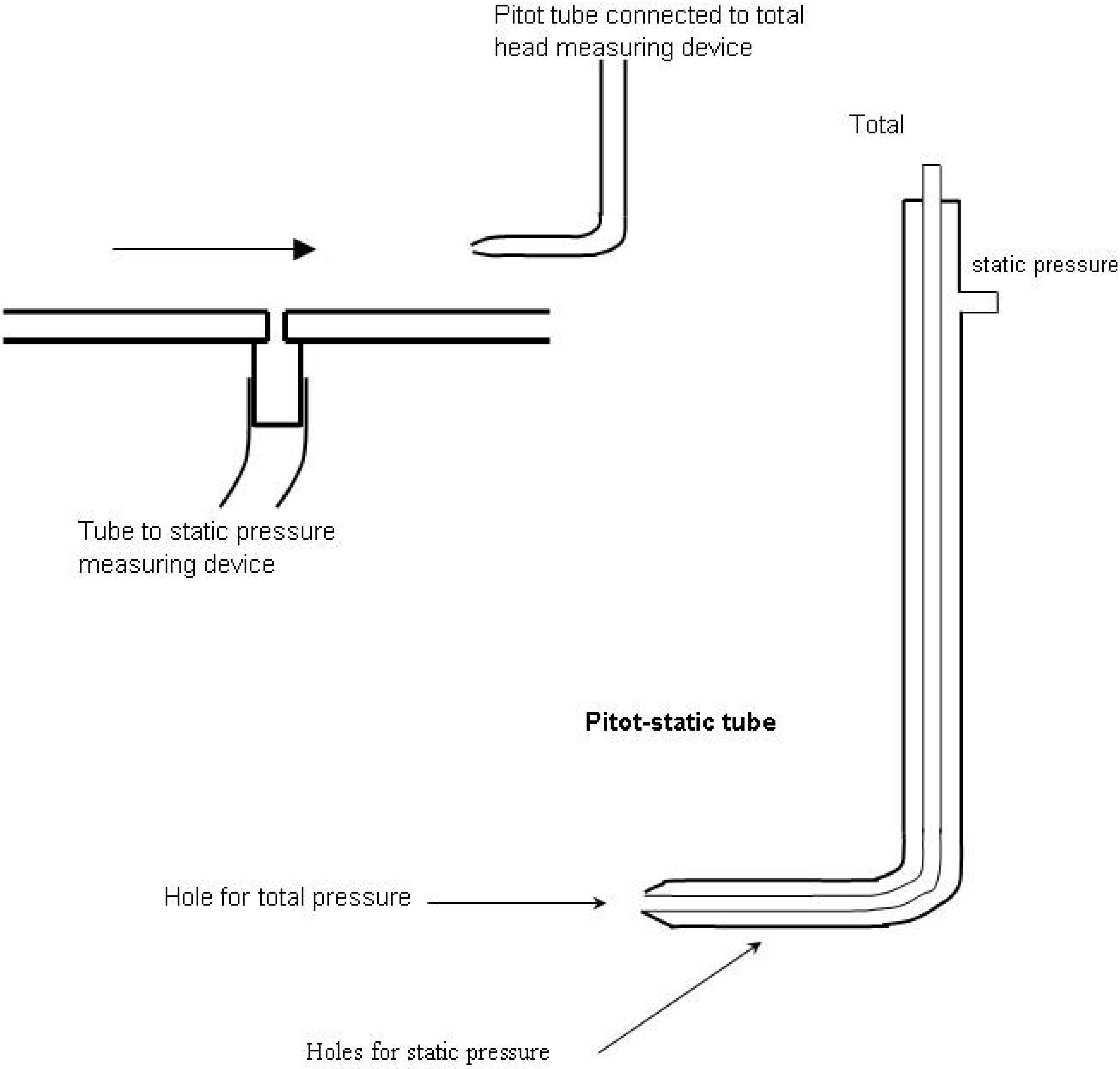
The sum **{**$p + \rho gz$**}** is often written as p^* - the piezometric pressure. We can then say:

$p^* + \frac{1}{2}\rho q^2$ **= constant along a streamline**

To measure the **static pressure** in a fluid flow, it is normal to make a small hole in the boundary wall of the flow and to connect the hole to a pressure measuring device - a manometer being the traditional instrument used.

To measure the **total pressure**, it is normal to employ a device known as a **Pitot tube**. This is a thin tube that can be pointed directly into the flow such that it is aligned exactly with the local streamlines. The other end of the tube is connected to a manometer (or other pressure measuring device). The streamline that meets the end of the tube within the flow is brought to rest - because there is no actual flow through the tube/manometer system - and therefore all the dynamic pressure is converted to static pressure. The sum of these two forms of static pressure is known as the **stagnation pressure** or total pressure.

To measure the **dynamic pressure**, the most common device (and the simplest and cheapest) used is a **Pitot-static tube**. This is a combination of the two techniques described above within one instrument. It consists of two thin concentric tubes bent into an L-shape; the inner tube has an open end which is pointed into the flow (as described above when measuring total pressure), while the outer tube is sealed and streamlined at its end but has a number of small holes around its circumference some way back from the end. The two tubes are connected across a differential pressure-measuring device (again, commonly a U-tube manometer), and the difference in pressure is the dynamic pressure.


Pitot tube connected to total head measuring device
Tube to static pressure measuring device
Total
static pressure
Pitot-static tube
Hole for total pressure
Holes for static pressure