Forces, centre of gravity, reactions and stability

Topic areas

Mechanical engineering:

- Centre of gravity
- Forces
- Moments
- Reactions
- Resolving forces on an inclined plane.

Mathematics:

- Angles
- Trigonometric identities

Prerequisites

It may be useful to look at the resource **'Centre of gravity of composite bodies'** to introduce the concepts of the centre of gravity and tipping.

Problem statement

Many engineered objects, such as buses, boats, diggers, pushchairs, furniture, etc. are required to be resistant to tipping over under normal use conditions. How can an engineer determine when an object will tip over?

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Background

The weight of an object is the force due to gravity acting between the earth and the object, and is measured in newtons (N). As the scale of the earth is large compared with many everyday objects, the weight force acting on an object, F, can be related to its mass, m, through the approximating expression

$$F = mg,$$

where g is the acceleration due to gravity, usually taken to be 9.81 ms^{-2} at the earth's surface.

When a force acts on a body the body will accelerate in the direction of the force unless there is a balancing force to oppose it.

For an object sitting on a plane, this balancing force is the normal reaction, R, which acts normally and upwards (or perpendicular) to the plane. If there is friction between the object and the plane which acts against any sliding tendencies, a frictional force acts upward along the plane.

Activity 1 – Discussion

Look at the following stationary block on a level plane (left) and an inclined plane (right).

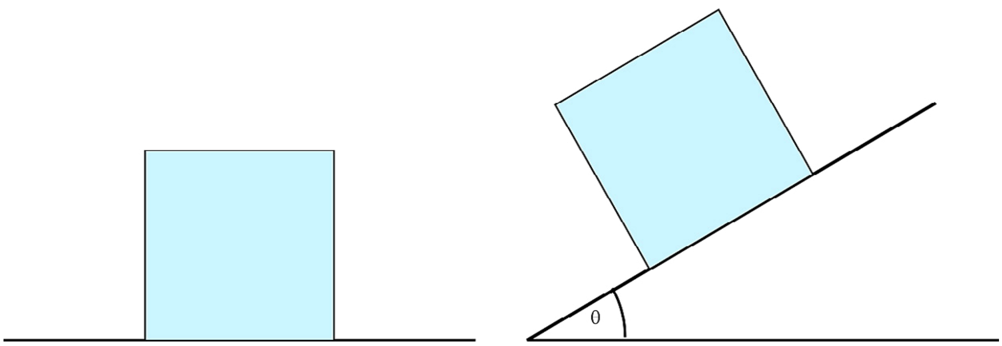


Figure 1
Stationary blocks

Discuss what forces are acting and draw force arrows showing where they are acting. What assumptions do you make?

Activity 2 - Stability

The resource tipping-1, shown in **Figure 2**, allows you to tilt an object with a uniform mass distribution and a mass of 1 kg. It is assumed that the friction between the object and the plane is large enough to prevent the object from sliding.

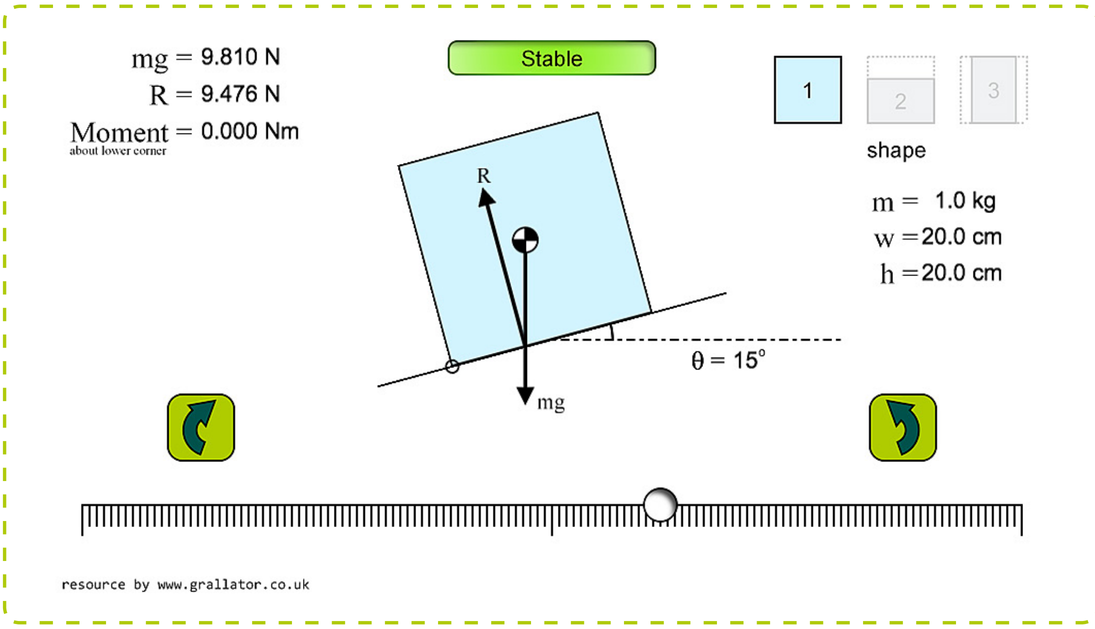
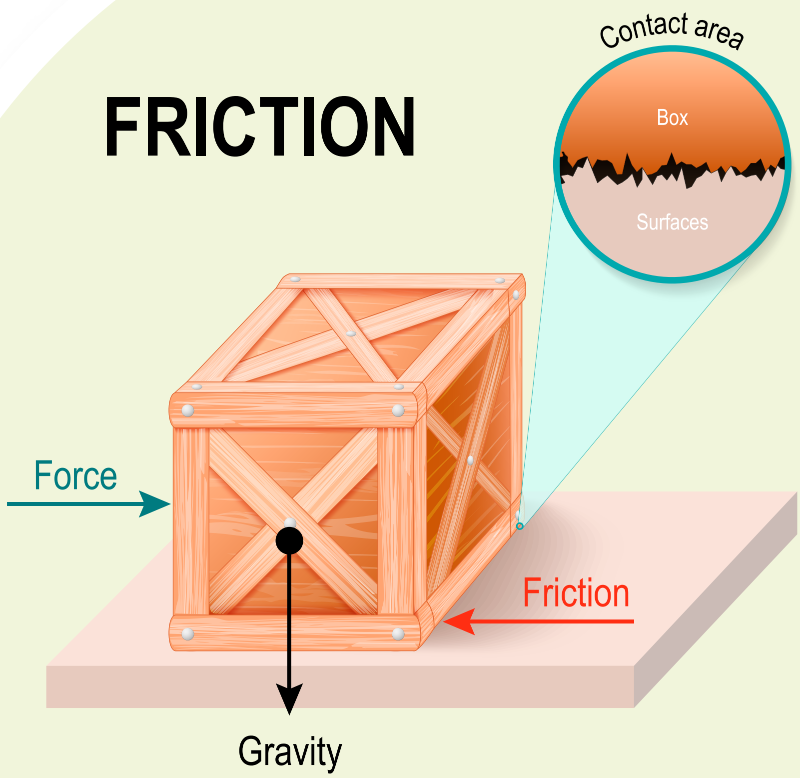


Figure 2
Screen shot of resource

The object can be tilted by dragging the slider, or clicking/tapping to buttons for fine control. Additionally, the object can be changed for a one of the same mass but different dimensions (the outline of the original square and original centre of gravity are also shown).

1. Investigate the angle at which shape 1 becomes unstable and wants to tip over. Why does it tip over?
2. Repeat the above with shapes 2 and 3. How does the shape affect the position of the centre of gravity and what does this do to the stability?



Background – Tipping angle

Many engineered objects, such as buses and pushchairs, require a physical tilt test to demonstrate adequate stability. However, it wouldn't be wise to build something as complex and expensive as a bus and then hope it passes such a test! During the design phase, engineers will have a good idea as to the expected stability by considering the location of the centre of gravity and where the limiting line of action lies. Indeed, many CAD packages will calculate the location of the centre of gravity.

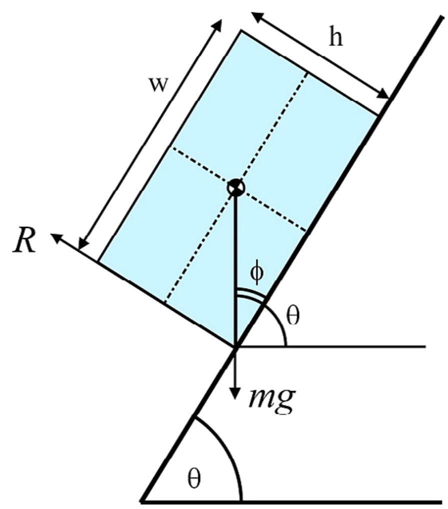


Figure 3
An object on a rough plane at the tipping point

The diagram in **Figure 3** shows a uniform block of width w and height h, so that the centre of gravity is at the centre, at the point of tipping. Note, the plane is rough so the object does not slide. The tipping point is reached when the weight force acts through the corner of the object. At this point, angles θ and ϕ sum to 90°.

$\theta + \phi = 90°$

The angle θ is the angle of the plane at which the object tips, i.e. $\theta = 90° - \phi$

The angle ϕ is the angle between the bottom edge and the line joining the corner and the centre of gravity, as shown in **Figure 10** in the Appendix (page 11).

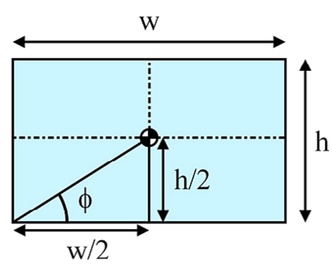


Figure 4

Using trigonometry

$$\tan\phi = \frac{h/2}{w/2} = \frac{h}{w}$$

so that the tipping angle is

$$\theta = 90^{\circ} - \tan^{-1}\left(\frac{h}{w}\right).$$

It can also be shown that (see Appendix)

$$\tan\theta = \frac{1}{\tan\phi} \Rightarrow \theta = \tan^{-1}\left(\frac{w}{h}\right).$$

Activity 3 - Finding the tipping angle

Calculate the tipping angle for the three objects considered in **Activity 2** and compare the results with the predictions of the resource **tipping-1**. The dimensions of the objects are given in **Table 1**.

Object	Width, w (cm)	Height, h (cm)	$\phi = \tan^{-1}\left(\frac{h}{w}\right)$	Calculated tipping angle θ	Tipping angle from interactive resource θ
1	20	20			
2	20	12			
3	12	20			

Table 1

Stretch and challenge activity

The resource **Bus1.html**, shown in **Figure 5**, allows you to tilt a bus that can be configured as

- A double decker (with and without upstairs passengers).
- A single decker
- Just the chassis

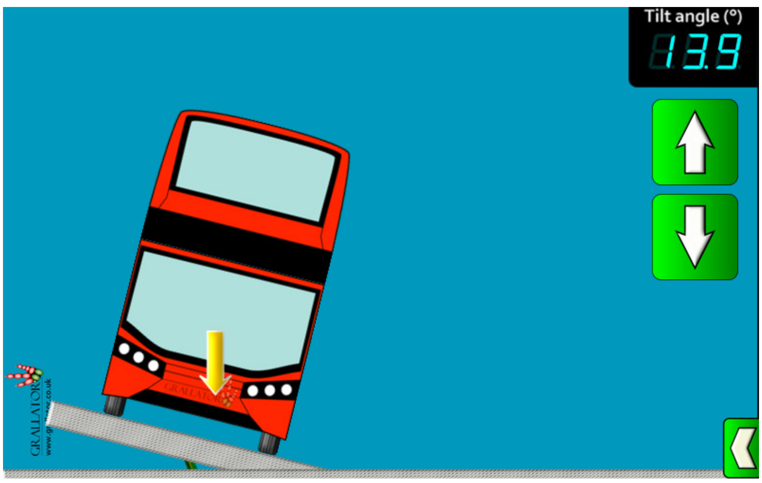


Figure 5
Screen shot of resource

The up and down arrows control the tilt angle of the bus. The left arrow at the bottom-right of the screen pulls out an options menu. The large yellow arrow shows the weight force acting from the centre of gravity of the bus.

Pull out the options menu and turn on the options for showing the tipping line, as shown in **Figure 6**.

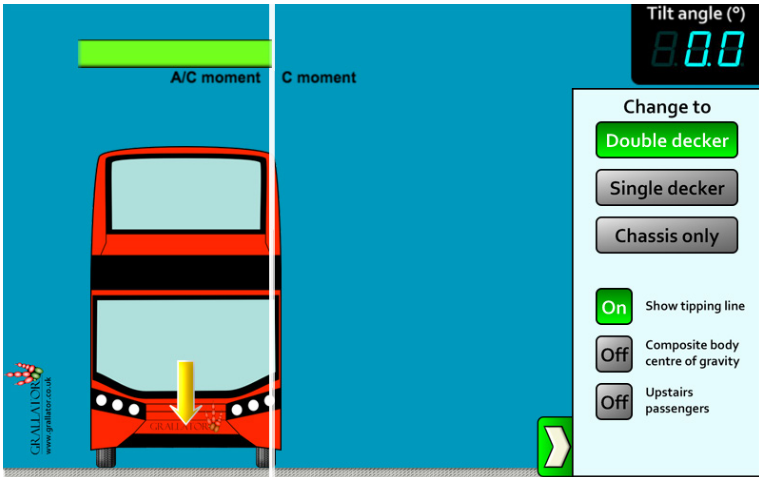


Figure 6
The options menu

1 A bus is made of a number of components such as

- A strong chassis that contains the heaviest items such as engine, cooling system, controls, suspension, wheels etc.
- The internal fittings such as the seats.
- The bodywork, which is usually made of relatively thin metal sheet.

Use the options menu to select the double decker, single decker and chassis only configurations and discuss the position of the centre of gravity of the bus for each.

2 Estimate the tipping angle for the three different configurations. Remember this will happen when the weight force acts outside the base of the object, which, in this case, is the wheel.

3 What effect does allowing for upstairs passengers have on the stability of the double decker configuration? (Check this by turning on the 'Upstairs passengers' option on the options menu.)

Notes and solutions

Activity 1

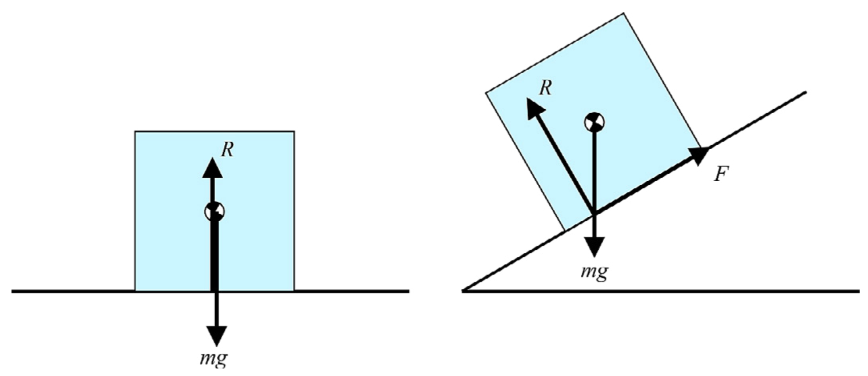


Figure 7
Force diagram

It is stated that the object is stationary, therefore there should be no net forces acting.

On the left-hand diagram, the weight force mg acts vertically downwards through the centre of gravity of the object. This is balanced by an equal and opposite normal reaction force, R. This force acts between the base of the object and the plane and through the centre of gravity.

On the right-hand diagram the weight force mg acts vertically downwards through the centre of gravity of the object. Again there is a normal reaction force, R, acting perpendicular to the plane. It is drawn acting at the base of the object where the line of action of the weight force intersects the base. As the object is stationary, there must also be a friction force acting along the plane to prevent sliding. This is also drawn on the base of the object acting from the same point as the normal reaction.

Activity 2 - Stability

1. The following observations are made

- Tilting the object moves the line of action of the normal reaction and the weight towards the corner.
- For angles up to 45° both the weight and the reaction fall within the base of the object there is no net moment about the lower corner of the object, and so it shows no tendency to tip, see **Figure 8**.

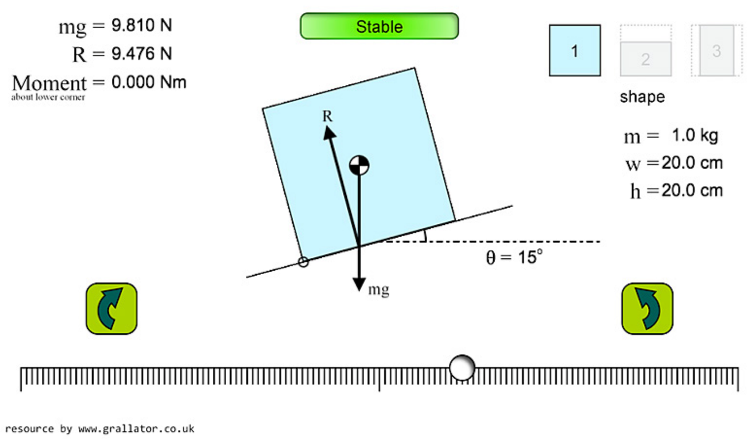


Figure 8
Screen shot of resource for a stable configuration

- For angles above 45°, the weight force acts outside the base of the object. However, the normal reaction cannot do this – it can only exert a force where it is in contact with the surface.
- When the normal reaction is concentrated at the corner and the weight force acts outside the base of the object there is a net turning moment and the object tips, see **Figure 9**.

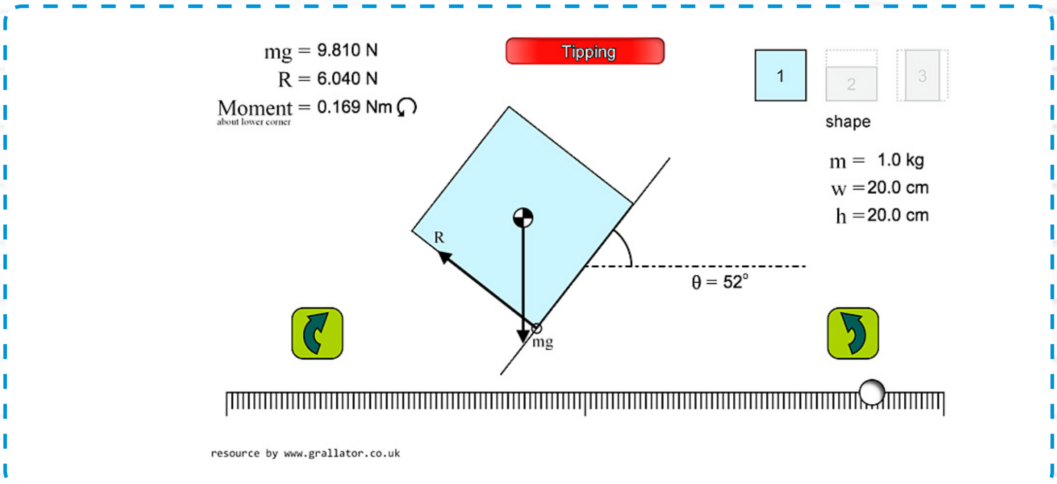


Figure 9
Screen shot of resource for an unstable configuration

2 The flatter object (shape 2) has the same mass as the square object, but as it is flatter, has a lower centre of gravity. The result of this is that the object can be tilted much more before it tips over.

The narrower object (shape 3), also of the same mass, has its centre of gravity at the same height as the square block. However, because it is narrower, the weight acts outside the base at a lower angle, therefore the angle of tilt required before tipping over is less than the square block.

Activity 3 - Finding the tipping angle

Values of the tipping angles for the shapes in **Table 1** are shown in **Table 2**.

Object	Width, w (cm)	Height, h (cm)	$\phi = \tan^{-1}\left(\frac{h}{w}\right)$	Calculated tipping angle θ	Tipping angle from interactive resource θ
1	**20**	**20**	45°	45°	46°
2	**20**	**12**	30.96° (2 d.p.)	59.04° (2 d.p.)	60°
3	**12**	**20**	59.04° (2 d.p.)	30.96° (2 d.p.)	31°

Table 2

A comparison shows that the interactive resource gives larger values than the calculation. For shapes 2 and 3 this can be explained by the fact that the resource only allows you to select integer values for the angle θ. The nearest integer below the calculated tipping angle will not tip while the nearest one above it will definitely tip.

For shape 1 there appears to be a disagreement. This is because the calculated value of θ represents the angle at which the object is on the point of tipping. Technically, at this angle the object will not tip (as asserted by the interactive resource), however, any infinitesimal increase in angle beyond this will lead to tipping. Hence θ is an upper limit before tipping starts.

Stretch and challenge activity

1. The bus is constructed using a strong chassis that contains the heaviest items such as engine, cooling system, controls, suspension, wheels etc. This has a centre of gravity reasonable close to the ground. On top of this sits the coachwork, including the seats. These add mass above the chassis and so raise the centre of gravity. However, the coachwork is light compared with the chassis so that the centre of gravity remains relatively low compared with the height of the bus, even when it is configured as a double decker. The individual contributions from the chassis, lower body work and upper body work can be viewed by turning on the "Composite body centre of gravity" option.

2. The bus will tip at the point where the line of action of the combined weight passes through the outer corner of the lower tyre.

   - For the chassis only configuration, the tipping angle is estimated to be about 56.6°.
   - For the single decker configuration the tipping angle is estimated to be about 50.8°.
   - For the double decker configuration the tipping angle is estimated to be about 38.6°.

3. When passengers are on the upper deck of a bus they add significant weight which moves the centre of gravity upwards. **Activity 3** showed that an object with a high centre of gravity is less stable than an object with a low centre of gravity. Adding upstairs passengers (use the pull-out menu option to do this) decreases the tipping angle to about 31.3°. In the UK it is legislation that a double decker which is fully loaded on the top deck bus must not tip when tilted at an angle of 28° (cited here **www.publications.parliament.uk/pa/cm198990/cmhansrd/1990-06-05/Debate-1.html**). The bus in this activity therefore passes!

Appendix

The main text states that it can also be shown that the tipping angle, θ is related to the angle ϕ through (refer to **Figures 3** and **4**).

$$\tan\theta = \frac{1}{\tan\phi} \Rightarrow \theta = \tan^{-1}\left(\frac{w}{h}\right).$$

This is an opportunity to use some angle formulae.

Approach 1

Taking the tangent of both sides of $\theta + \phi = 90°$ gives $\tan(\theta + \phi) = \tan 90°$. Using the angle sum formula for tan

$$\tan(\theta+\phi) = \tan 90^{\circ}$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tan 90^{\circ}$$

The right hand side may be considered to be a problem – a calculator will give an error if you try to find the tan of 90°. However, if you look at a plot of tan θ, you will see that tan $\theta \rightarrow \infty$ as $\theta \rightarrow 90°$.

Rearranging the above

$$\frac{\tan\theta + \tan\phi}{\tan 90^{\circ}} = 1 - \tan\theta\tan\phi$$

As tan 90° is infinitely large, any finite number divided by this is zero, so that

$$\frac{\tan\theta + \tan\phi}{\tan 90^{\circ}} = 0 = 1 - \tan\theta\tan\phi$$

$\Rightarrow$

$$1 = \tan\theta\tan\phi$$

$$\tan\theta = \frac{1}{\tan\phi}$$

Substituting $\tan\phi = \frac{h}{w}$ gives

$$\tan\theta = \frac{w}{h}$$

Approach 2

Consider the shape at a general angle as shown in **Figure 10**.

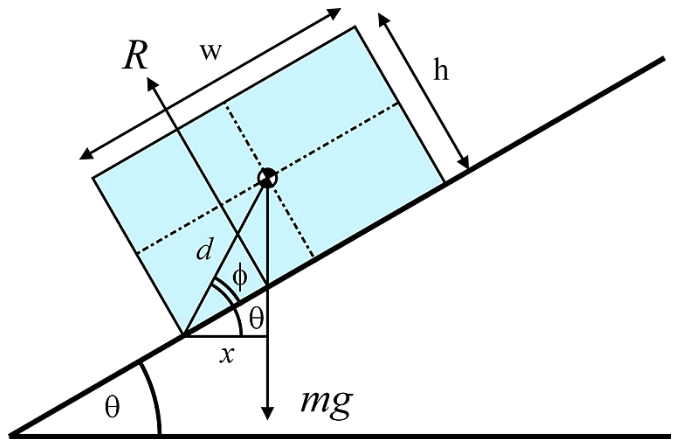


Figure 10

The horizontal distance from the corner to the line of action of the weight force is denoted x in the diagram. The distance from the corner to the centre of gravity is denoted d. The angle θ is the tilt angle and ϕ is the angle between the bottom edge and the line joining the corner and the centre of gravity. Its value is given by $\tan\phi = \frac{h/2}{w/2} = \frac{h}{w}$ (see solution for Activity 3, above).

Using trigonometry

$$x = d\cos(\theta+\phi).$$

At the tipping point $x = 0$ as the line of action of the weight passes through the corner.
Using this and the angle sum formula for cos

$$\begin{aligned}
0 &= d\cos(\theta+\phi) \\
0 &= \cos(\theta+\phi) \qquad (d \neq 0) \\
0 &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\sin\theta\sin\phi &= \cos\theta\cos\phi \\
\frac{\sin\theta}{\cos\theta} &= \frac{\cos\phi}{\sin\phi} \\
\tan\theta &= \frac{1}{\tan\phi} = \frac{w}{h}
\end{aligned}$$

i.e.

$$\tan\theta = \frac{w}{h}$$

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