Compressive Sensing – A 25 Minute Tour

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Dedication

Dennis Healy, 1957 – 2009
Compressive sensing today

Enormous field spanning
- mathematics
- applied mathematics
- computer science
- information theory
- signal processing
- circuit design
- optical engineering
- biomedical imaging
- ...

Many contributors; e.g. R. Baraniuk

http://nuit-blanche.blogspot.com/
http://dsp.rice.edu/cs

Our focus: short and biased overview
A contemporary paradox

- Massive data acquisition
- Most of the data is redundant and can be thrown away
- Seems enormously wasteful
Massive data acquisition

Most of the data is redundant and can be thrown away

Seems enormously wasteful

One can regard the possibility of digital compression as a failure of sensor design. If it is possible to compress measured data, one might argue that too many measurements were taken.

David Brady
Going against a long established tradition?

- Acquire/Sample (A-to-D converter, digital camera)
- Compress (signal dependent, nonlinear)

Fundamental question
Can we directly acquire just the useful part of the signal?
What Is Compressive Sensing?
In a nutshell...

- Can obtain super-resolved signals from just a few sensors
- Sensing is *nonadaptive*: no effort to understand the signal
- Simple acquisition process followed by numerical optimization

First papers
- Candès, Romberg and Tao, 2006
- Candès and Tao, 2006
- Donoho, 2006

By now, very rich mathematical theory
Sparsity: wavelets and images

1 megapixel image
Sparsity: wavelets and images

1 megapixel image

wavelet coefficients

zoom in
Implication of sparsity: image “compression”

1. Compute 1,000,000 wavelet coefficients of mega-pixel image
2. Set to zero all but the 25,000 largest coefficients
3. Invert the wavelet transform

original image
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This principle underlies modern lossy coders (sound, still-picture, video)
Idealized sampling

- $x$: signal coefficients in our convenient representation
- Collect information by measuring largest components of $x$

![Diagram showing sparse and nearly sparse signals](image-url)
Idealized sampling

- $x$: signal coefficients in our convenient representation
- collect information by measuring largest components of $x$

What if these positions are not known in advance?
- what should we measure?
- how should we reconstruct?
Incoherent/random sensing

\[ y = \langle a_k, x \rangle, \quad k = 1, \ldots, m \]

- Want sensing waveforms as spread out/“incoherent” as possible
- Span of \( \{a_k\} \) should be as random as possible (general orientation)

\[ a_k \overset{\text{i.i.d.}}{\sim} F \]

\[ \mathbb{E} a_k a_k^* = I \] and \( a_k \) spread out
- \( a_k \) i.i.d. \( \mathcal{N}(0, 1) \) (white noise)
- \( a_k \) i.i.d. \( \pm 1 \)
- \( a_k = \exp(i2\pi\omega_k t) \) with i.i.d. frequencies \( \omega_k \)
- ...
Incoherence

- Signal is local, measurements are global
- Each measurement picks up a little information about each component
Example of foundational result

Classical viewpoint

- Measure everything (all the pixels, all the coefficients)
- Keep $d$ largest coefficients: distortion is $\|x - x_d\|$
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Classical viewpoint
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Compressed sensing viewpoint
- Take $m$ random measurements: $y_k = \langle x, a_k \rangle$
- Reconstruct by linear programming: ($\|x\|_{\ell_1} = \sum_i |x_i|$)

$$x^* = \arg \min \|\tilde{x}\|_{\ell_1} \text{ subject to } y_k = \langle \tilde{x}, a_k \rangle, \ k = 1, \ldots, m$$

Among all the objects consistent with data, pick min $\ell_1$
Example of foundational result

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Same performance with about $m = d \log n/d$ (sketch)

$$\|x^* - x\|_{\ell_2} \leq \|x - x_d\|_{\ell_2}$$
Example

- Take 96K incoherent measurements of “compressed” image
- Compressed image is perfectly sparse (25K nonzero wavelet coeff)
- Solve $\ell_1$

original (25k wavelets)
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- Compressed image is perfectly sparse (25K nonzero wavelet coeffs)
- Solve $\ell_1$

original (25k wavelets)  perfect recovery
What is compressive sensing?

Possibility of compressed data acquisition protocols which directly acquire just the important information

- Incoherent/random measurements $\rightarrow$ compressed description
- Simultaneous signal acquisition and compression!

All we need is to decompress…
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Three surprises

- Sensing is ultra efficient and nonadaptive
- Recovery is possible by tractable optimization
- Sensing/recovery is robust to noise (and other imperfections)
Applications and Opportunities
Three potentially impacted areas

- Analog-to-digital conversion
- Optical systems
- Magnetic Resonance Imaging
Time or space sampling

- Analog-to-digital converters, receivers, ...
- Space: cameras, medical imaging devices, ...

Nyquist/Shannon foundation: must sample at twice the highest frequency
Sampling of ultra wideband radio frequency signals
Hardware brick wall

- Signals are wider and wider band
- “Moore’s law:” factor 2 improvement every 6 to 8 years

Extremely fast/high-resolution samplers are decades away
Hardware brick wall

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Way out? Analog-to-information (DARPA)
What have we learned?

If ‘information bandwidth’ less than total bandwidth, then should be able to
- sample below Nyquist without information loss
- recover missing samples by convex optimization
New analog-to-digital converters

Joint with Caltech and Northrop Grumman

Manufactured three chips

- Nonuniform sampler (InP)
- Random modulator pre-integration
  - 4 channels (InP)
  - 8 channels (CMOS)
Random modulator pre-integrator (RMPI)

Joint with Becker, Emami and Yoo (Caltech)

RMPI Basic architecture

- Nyquist rate 5GHz
- 8 channels at 50MHz \( \rightarrow \) 400MHz
- 12.5\times undersampled
RMPI chip v.2

Process:
IBM 90nm CMOS9SF 06_02_00_LB

- Power Consumption: 830mW
- Bandwidth: 2.5GHz
- Dynamic Range: 50dB
- Sampling rate: 400 MS/s
- Die Area: 2mm × 2mm
Pulse recovery from full system simulations

Frequency domain. Two overlapping pulses at 103.3 MHz and 2410.6 MHz carriers
Amplitude 1: 3.2e–03, Amplitude 2: 1.0e–02 (10.0 dB dynamic range)
N=2048 (T=410 ns), jitter rms = 0.5 ps
Estimate pulse 1 to have frequency 2410.6 MHz (true value 2410.6 MHz)
Estimate pulse 2 to have frequency 103.3 MHz (true value 103.3 MHz)

Time domain., MSE=6.2 \times 10^{-3}
Rel. $l_2$ error: 7.8 \times 10^{-2}; rel. $l_\infty$ error: 1.3 \times 10^{-1}; rel. $l_\infty$ freq error: 5.0 \times 10^{-2}
MSE on pulse envelopes: 5.0 \times 10^{-3}, 1.9 \times 10^{-3}

TOA estimate: 66.6 ns, Actual: 67.0 ns, Error: 0.4 ns
Duration estimate: 203.4 ns, Actual: 199.8 ns, Error: 3.6 ns
TOA estimate: 107.2 ns, Actual: 110.2 ns, Error: 3.0 ns
Duration estimate: 206.2 ns, Actual: 200.0 ns, Error: 6.2 ns
Pulse recovery from full system simulations

Frequency domain. Two overlapping pulses at 703.3 MHz and 2103.6 MHz carriers
Amplitude 1: 1.0e−02, Amplitude 2: 1.0e−02 (0.0 dB dynamic range)
N=2048 (T=410 ns), jitter rms = 0.5 ps
Estimate pulse 1 to have frequency 2103.6 MHz (true value 2103.6 MHz)
Estimate pulse 2 to have frequency 703.3 MHz (true value 703.3 MHz)

Time domain., MSE=1.1⋅10^{-3}
Rel. l_2 error: 3.4⋅10^{-2}; rel. l_\infty error: 5.4⋅10^{-2}; rel. l_\infty freq error: 2.6⋅10^{-2}
MSE on pulse envelopes: 1.5⋅10^{-3}, 2.6⋅10^{-4}

TOA estimate: 66.6 ns, Actual: 67.0 ns, Error: 0.4 ns
Duration estimate: 200.8 ns, Actual: 199.8 ns, Error: 1.0 ns
TOA estimate: 107.2 ns, Actual: 110.2 ns, Error: 3.0 ns
Duration estimate: 204.2 ns, Actual: 200.0 ns, Error: 4.2 ns
Real testing: NGC-Caltech A-to-I Receiver Test

(Die at similar scale)
Real results

RMPI (InP) behaves as simulated
- 2.5GHz of bandwidth
- 50-60dB of dynamic range
- $\sim 3W$ of power consumption

Other ADC efforts
- Eldar et al. (Technion)
- Fudge et al. (L3 & Wisconsin)
- Baraniuk et al. (Rice)
Three potentially impacted areas

- Analog-to-digital conversion
- Optical systems
- Magnetic Resonance Imaging
What do we measure?

- **Direct sampling**: analog/digital photography, mid 19th century

- **Indirect sampling**: acquisition in a transformed domain, second half of 20th century; e.g. CT, MRI

- **Compressive sampling**: acquisition in an incoherent domain
  - Design incoherent analog sensors rather than usual pixels
  - Pay-off: need far fewer sensors

*The first photograph?*

*CT scanner*
One pixel camera

Richard Baraniuk, Kevin Kelly, Yehia Massoud, Don Johnson
Rice University, dsp.rice.edu/CS
MIT Tech review: top 10 emerging technologies for 2007

Other works: Brady et al., Freeman et al., Wagner et al., Coifman et al.
"Single-Pixel" CS Camera

Kevin Kelly
Richard Baraniuk
Rice University

scene

$\mathbf{x}$

single photon detector

random pattern on DMD array (spatial light modulator)

$A_j$

DMD

$y$

can be exotic

• IR, UV, THz, PMT, spectrometer, ...

color target

raster scan IR

CS IR

hyperspectral data cube
Three potentially impacted areas

- Analog-to-digital conversion
- Optical systems
- Magnetic Resonance Imaging
Fast Magnetic Resonance Imaging

Goal: sample less to speed up MR imaging process

- Fully sampled
- $6 \times$ undersampled classical
- $6 \times$ undersampled CS

*Trzasko, Manduca, Borisch (Mayo Clinic)*
MR angiography

- Fully sampled
- 6 × undersampled

Trzasko, Manduca, Borisch
Trzasko, Manduca, Borisch
Compressive sensing in the news

Wired, March 2010
Breath-hold post-gadolinium MRA in a 9 year old male with hypertension using 4X acceleration at 1.2 mm3 resolution. Left images (a, c) are with ARC and right (b, d) are with L1-SPIRiT compressed sensing. Note improved delineation of pancreas (big arrow), pancreatic duct (middle arrow), and diaphragm (small arrow) with $\ell^1$-SPIRiT. Left gastric artery (arrowhead) emerges from the noise.
Submillimeter near-isotropic resolution MRI in an 8-year-old male. Post-contrast T1 imaging with an acceleration of 4. Standard (a, c) and compressed sensing reconstruction (b, d) show improved delineation of the pancreatic duct (vertical arrow), bowel (horizontal arrow), and gallbladder wall (arrowhead) with L1-SPIRiT reconstruction, and equivalent definition of the portal vein (black arrow).
Summary

Ultra efficient acquisition protocol:

*automatically translates analog data into already compressed digital form*

- Change the way engineers think about data acquisition
- Already many applications
- More applications to come