Study of the Motion of Knee and Ankle during Cycling

Public Health Engineering, Sports Technology

Introduction

Staying healthy means eating the right things and taking plenty of exercise. However, vigorous exercise, if not undertaken carefully, can result in serious injury. This exemplar uses mathematics to compare the potential for injury to the knee or ankle when cycling and when jogging. To make good use of this exemplar, you should first know that humans have evolved good resistance to compressive forces on knee and ankle joints as these occur naturally, but not good resistance to the extension forces that can sometimes occur when cycling.

Forces on the Human Body when Cycling

Figure-1 shows the layout of a human leg when cycling. For simplicity, we shall assume that the vertical axis is x-axis and horizontal axis is y-axis. Also, it is assumed that the knee moves only up and down a distance x_k whilst the ankle describes a perfect circle of radius r moving with an angular velocity ω and making an angle ωt with the vertical axis at any time t.

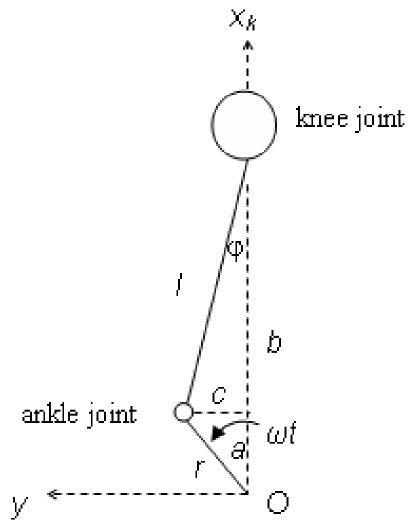

Figure-1: Layout of a human leg when cycling

By inspection of Figure-1, the following relationship is found:

$$x_k(t) = a + b = r\cos\omega t + l\cos\phi \quad \ldots (1)$$

Remembering that $(\cos\phi)^2 = 1 - (\sin\phi)^2$, we can write:

$$\cos\phi = \sqrt{1 - (\sin\phi)^2} \quad \ldots (2)$$

Hence equation (1) can be rewritten as:

$$x_k(t) = r\cos\omega t + l\left(1 - (\sin\phi)^2\right)^{1/2} \quad \ldots (3)$$

Also, using simple trigonometry in two triangles formed, it can be written that:

$$r\sin\omega t = c \quad \ldots (4)$$

$$l\sin\phi = c \quad \ldots (5)$$

Equating (4) and (5), we get:

$$r\sin\omega t = l\sin\phi \text{, or}$$

$$\sin\phi = \frac{r}{l}\sin\omega t \quad \ldots (6)$$

Using this equation (6), we can finally write equation (3) representing the displacement of knee in y direction as:

$$x_k(t) = r\cos\omega t + l\sqrt{1 - \left(\frac{r}{l}\sin\omega t\right)^2} \quad \ldots (7)$$

The term under the square root can be expanded using the binomial theorem:

$$\left(1 - \left(\frac{r}{l}\sin\omega t\right)^2\right)^{1/2} = \left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{1/2}$$

$$= 1^{1/2} + \frac{1}{2}(1)^{-1/2}\left(-\frac{r^2}{l^2}\sin^2\omega t\right)$$

$$+ \left(\frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\right)(1)^{-1.5}\left(-\frac{r^2}{l^2}\sin^2\omega t\right)^2 + \ldots$$

The expression $\left(-\frac{r^2}{l^2}\sin^2\omega t\right)$ in the above expansion is a very small number (*please see Extension Activity-1*) and hence its higher powers will be even smaller. Therefore, neglecting all but the first two terms, we get:

$$\left(1 - \left(\frac{r}{l}\sin\omega t\right)^2\right)^{1/2} = 1 - \frac{r^2}{2l^2}\sin^2\omega t \quad \ldots (8)$$

Using the standard trigonometry relation $\cos 2\omega t = 1 - 2\sin^2\omega t$, we can write

$$\sin^2\omega t = \frac{1 - \cos 2\omega t}{2}$$

Hence, equation (8) becomes:

$$\left(1 - \left(\frac{r}{l}\sin\omega t\right)^2\right)^{1/2} = 1 - \frac{r^2}{2l^2}\left(\frac{1 - \cos 2\omega t}{2}\right)$$

or,

$$\left(1-\left(\frac{r}{l}\sin\omega t\right)^2\right)^{\frac{1}{2}} = 1-\frac{r^2}{4l^2}+\frac{r^2}{4l^2}\cos 2\omega t \quad \ldots (9)$$

Using this new value of the square root term, equation (7) can now be written as:

$$x_k(t) = r\cos\omega t + l - \frac{r^2}{4l} + \frac{r^2}{4l}\cos 2\omega t \text{ , or}$$

$$x_k(t) = l - \frac{r^2}{4l} + r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \quad \ldots (10)$$

Since it is assumed that the knee does not have any movement in the y direction, we only have one component in its displacement and that is in the x direction only.

Inertia Forces

Inertia force is a force that arises due to non-uniform motion relative to two different frames of reference. There is an inertia force on the knee joint caused by the accelerations of the knee and the mass of the lower leg. This can cause injury to the knee when accelerations are large: for instance when changing to a low gear when cycling quickly.

For this simplified modelling of the inertia forces, the mass of the lower leg will be distributed as shown in Figure-2 for a considerable simplification in the calculation of the inertia forces. The centre of mass of the lower leg is somewhere between the knee and the foot (shown as big black dot) and its motion is actually more complex than that described using the geometry shown above in Figure-1. In addition, the true limb has a particular moment of inertia, and this cannot be replicated accurately when the lower leg mass is distributed. So, it should be remembered that the following analysis only yields approximate values for the knee and ankle forces caused by the inertia of the lower leg.

The following relationships can be written directly from Figure-2:

$$m_{leg} = m_k + m_a \quad \ldots (11)$$

$$m_a l_a = m_k l_k \quad \ldots (12)$$

We can solve equations (11) and (12) simultaneously. From equation (11), we can write:

$$m_k = m_{leg} - m_a$$

Substituting this value in equation (12), we get:

$$m_a l_a = m_{leg} l_k - m_a l_k$$

$$\text{or, } m_a(l_a + l_k) = m_{leg} l_k$$

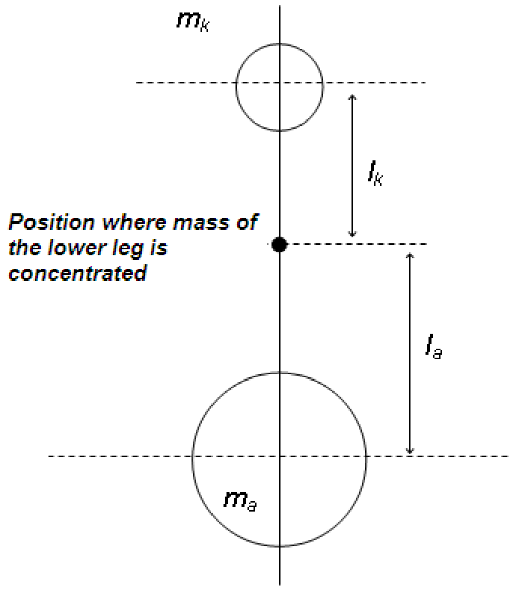

Figure-2: Distribution of effective mass of the lower leg

Finally, by rearranging the terms, we get:

$$m_a = m_{leg}\left(\frac{l_k}{l_a + l_k}\right) \quad \ldots (13)$$

Similarly, we can derive

$$m_k = m_{leg}\left(\frac{l_a}{l_a + l_k}\right) \quad \ldots (14)$$

We will assume that two thirds of the lower leg mass is concentrated at the ankle end of the leg and one third at the knee end. The one disadvantage with this analysis so far is that there is no attempt to ensure that the leg rotational inertia is correctly described.

The rotation of the ankle mass m_a is given by

$$x_a(t) = r\cos\omega t\,\bar{i} + r\sin\omega t\,\bar{j} \quad \ldots (15)$$

where r is the radius of the circle made by the ankle on the pedal crank, $\bar{i}$ is the unit vector in the x direction and $\bar{j}$ is the unit vector in the y direction. Differentiating equation (15) with respect to time gives the velocity of the ankle:

$$v_a(t) = -r\omega\sin\omega t\,\bar{i} + r\omega\cos\omega t\,\bar{j} \quad \ldots (16)$$

Assuming that $v_a(t)$ is kept constant in the analysis, differentiating equation (16) with respect to time again gives the acceleration of the ankle:

$$a_a(t) = -r\omega^2\cos\omega t\,\bar{i} - r\omega^2\sin\omega t\,\bar{j} \quad \ldots (17)$$

Remember that the linear displacement of the knee is given by equation (10). Differentiating it with respect to time gives the velocity of the knee:

$$v_k(t) = -r\omega\sin\omega t - \frac{r^2}{2l}\omega\sin 2\omega t \quad \ldots (18)$$

Differentiating equation (18) again gives the acceleration of the knee:

$$a_k(t) = -r\omega^2 \cos\omega t - \frac{r^2\omega^2}{l} \cos 2\omega t \quad \ldots (19)$$

Please note that the quantities in equation (18) and (19) are vector quantities with their y-component being zero.

The rotational acceleration of the ankle is now known (equation 17) as well as the linear acceleration of the knee mass (equation 19). Remembering Newton's second law of motion which can be expressed as:

$$\text{Force} = \text{Mass} \times \text{Acceleration},$$

the magnitude of the total inertia force in the x direction can be written by taking only x components from equation (17) and (19), and we get:

$$F_x = (m_a)(-r\omega^2 \cos\omega t) + (m_k)\left(-r\omega^2 \cos\omega t - \frac{r^2\omega^2}{l}\cos 2\omega t\right)$$

or

$$F_x = -(m_a + m_k)(r\omega^2 \cos\omega t) - (m_k)\left(\frac{r^2\omega^2}{l}\cos 2\omega t\right) \quad \ldots (20)$$

Similarly, the total inertia force in the y direction can be written by collecting only y components from equation (17) and (19), and we get:

$$F_y = -m_a r\omega^2 \sin\omega t \quad \ldots (21)$$

Hence, the total inertia force can be written as:

$$F = F_x\bar{i} + F_y\bar{j} \quad \ldots (22)$$

where the magnitude of the force can be calculated as:

$$F = |F| = \sqrt{F_x^2 + F_y^2} \quad \ldots (23)$$

Sample Calculations

Consider the following data representing average values for a male:

Length of lower leg, l = 40 cm = 0.4 m

Total mass of lower leg, m_{leg} = 4.88 kg

Mass centred on ankle,

$$m_a = \frac{2}{3} \times m_{leg} = \frac{2}{3} \times 4.88 = 3.25 \text{ kg}$$

Mass centred on knee,

$$m_k = \frac{1}{3} \times m_{leg} = \frac{1}{3} \times 4.88 = 1.63 \text{ kg}$$

Radius of circle made by ankle during cycling,

r = 18 cm (say) = 0.18 m

If we assume that the pedalling rate is one revolution per second, the angular velocity of the ankle will be $\omega = 2\pi$ radian/sec.

Please observe that for this angular velocity,

$$\sin 2n\pi = 0, \text{for all } n$$

$$\cos 2n\pi = 1, \text{for all } n$$

Substituting these values in equation (20), we get

$$F_x = -4.88\left(0.18 \times (2\pi)^2 \times \cos 2\pi t\right) - 1.63\left(\frac{(0.18)^2 \times (2\pi)^2}{0.4}\cos 4\pi t\right)$$

$$\Rightarrow F_x = -39.85 \approx -40 \text{ N}$$

And substituting these values in equation (21), we get

$$F_y = 0$$

Finally, using equation (23), we find that the magnitude of the inertial force on the ankle will be approximately 40 N (the negative sign shows that the force is working downward). But, if you observe equation (20), it is evident that the force is directly proportional to the square of the angular velocity ω, i.e., the force will increase with the increase in this angular velocity. So, if a cyclist, who is travelling at considerable speed, were to suddenly drop into a very low gear this force could rise by a factor of 5-10. This increased force will be enough to cause serious damage to the knee ligaments that are not evolved to withstand sudden application of several hundred Newton force in extension.

Comparison with Jogging

The inertia forces in the vertical direction on the lower leg and its joints are relatively easy to estimate. The inertia mass is equal to the total mass of the human body. The acceleration a of the ankle can be calculated directly from the general equation of motion:

$$v^2 = u^2 + 2ax \quad \ldots (26)$$

where u is the velocity of the moving foot in the vertical direction when it hits the ground, v is zero at the instant the ankle comes to rest on the ground and x is the deflection of the sole of the shoe as it absorbs the impact of the foot on the ground (assuming the ground is unyielding).

Sample Calculations

Consider the following data representing average values for a male:

Total average mass of a male body, m = 75 kg

Also,

v =0;

u = 2 m/sec (an estimate only);

x = –2 cm = –0.02 m

Thus, acceleration a of the ankle is:

$$a = \frac{v^2 - u^2}{2x} = \frac{0 - 4}{2 \times (-0.02)} = 100 \text{ m/sec}^2$$

Hence, force can be calculated as:

$$\text{Force} = \text{Mass} \times \text{Acceleration},$$

$$\text{Force} = 75 \times 100 = 7500\,\text{N} = 7.5\,\text{kN}$$

The force transmitted through the ankle when it strikes the ground is thus given by 7.5 kN. This surprisingly high force is in fact reduced somewhat as the cartilage within the ankle, knee and hip joints also provides some additional cushioning (i.e. increases x in practice). But, as a first approximation, it shows why impact loads can be so high while jogging on a 'hard' (i.e. unyielding) surface like concrete. This impact load can be reduced by running on 'soft' surfaces like mud or sand. It can also be reduced by increasing the thickness of the elastic sole on your trainers. Fortunately however, knee and ankle joints have evolved to withstand very high compressive forces (unlike the extension forces caused in the cycling example above).

Conclusion

The calculations and discussion in this exemplar lead us to conclude that cycling causes less compressive force on knee and ankle joints than jogging, but because sudden and significant extension forces can be generated in cycling, there is still the potential for injury.

Also, the information given in this exemplar may prove to be useful to an engineer working in the Sports Technology sector to offer new ideas to develop knee and ankle protection devices. It may also be useful to those designing artificial joints that could be used to replace damaged knee and ankle joints.

Extension Activity – 1:

Using the figures used in the sample calculations, show that the term $\left(-\frac{r^2}{l^2}\sin^2\omega t\right)$ used in equation (7) has a very small value compared to the other terms.

Extension Activity – 2:

The female equivalent anthropometric values are about 92% of the male values used above. Discuss the impact of forces in the case of a female in a similar manner.

Where to find more

1. *Basic Engineering Mathematics*, John Bird, 2007, published by Elsevier Ltd.
2. *Engineering Mathematics*, Fifth Edition, John Bird, 2007, published by Elsevier Ltd.

Cindy Carlson – Public Health and Institutional Development Specialist

Cindy has worked in public health and programme management in the UK and in developing countries for the last 24 years. She is keen to see how different disciplines contribute improving people's health and spent a number of years overseeing water supply programmes that were managed and implemented by engineers in Africa. Her UK work has looked particularly at how people working in engineering and the built environment could do more to support health improvement initiatives.

Information for Teachers

The teachers should have some knowledge of

- Binomial Theorem
- Inertia forces
- Newton's law of motion
- Differentiation
- Components of vector quantities

Topics covered from "Mathematics for Engineering"

- Topic 1: Mathematical Models in Engineering
- Topic 3: Models of Oscillations
- Topic 6: Differentiation and Integration
- Topic 7: Linear Algebra and Algebraic Processes

Learning Outcomes

- LO 01: Understand the idea of mathematical modelling
- LO 03: Understand the use of trigonometry to model situations involving oscillations
- LO 06: Know how to use differentiation and integration in the context of engineering analysis and problem solving
- LO 07: Understand the methods of linear algebra and know how to use algebraic processes
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

Assessment Criteria

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 3.1: Solve problems in engineering requiring knowledge of trigonometric functions
- AC 3.2: Relate trigonometrical expressions to situations involving oscillations
- AC 6.1: Calculate the rate of change of a function
- AC 7.1: Solve engineering problems using vector methods
- AC 7.3: Solve problems involving arithmetic and geometric sequences and series
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

Links to other Units of The Advanced Diploma in Engineering

- Unit-1: Investigating Engineering Business and the Environment
- Unit-8: Mathematical Techniques and Applications for Engineers
- Unit-9: Principles and Application of Engineering Science