

INTRODUCTION

On process plants, tanks are used to store large quantities of liquids that can be very hazardous and poisonous. In some situations, it may be necessary to remove the liquid contents of these tanks as quickly as possible. In other cases, the liquid in the tank may be prone to contamination, and so the tank has to be drained as quickly as possible.

When designing the tank, a small blow down orifice is built in next to a valve, which when opened allows the tank to be drained. It is important that the orifice is large enough so that the tank is emptied sufficiently quickly, but is not too large since this will lead to flooding of the drainage system.

This exemplar shows how to build a mathematical model of liquid draining through a tank and how to use the model to determine the time required for a tank to completely drain.

MATHEMATICAL MODEL

Consider a cylindrical tank, with a diameter D , partially filled with a liquid to a height h . The vapour pressure in the tank is constant at P_A . The tank is emptying through an orifice of diameter d at the base of the tank to a drain of pressure P_B .

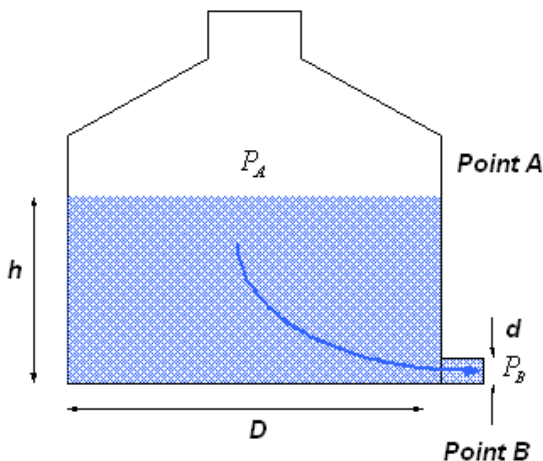


Figure-1: 2D view of the tank filled with liquid

If the flow can be assumed to be *steady* (i.e. not varying with time) and *inviscid* (i.e. forces and losses due to the viscosity of the fluid are negligible) and the fluid is *incompressible* (i.e. its density does not vary), Bernoulli's conservation of energy equation states that at any point within the flow of fluid:

$$P + \frac{1}{2} \rho v^2 + \rho gz = \text{constant} \dots (1)$$

where P is the pressure, ρ is the density of the fluid, v is the velocity, z is the elevation, and g is the gravitational acceleration.

In Figure-1, the flow of the liquid is from point A to point B. In this case, the flow is not actually steady because the height of the surface varies with time, but, if d is small compared to D , h will change relatively slowly, and we can treat the flow as being approximately steady for the purposes of flow analysis without too much loss of accuracy. Then according to the Bernoulli's equation, we get:

$$P_A + \frac{1}{2} \rho v_A^2 + \rho gz_A = P_B + \frac{1}{2} \rho v_B^2 + \rho gz_B \dots (2)$$

Here, v_A is the velocity of the fluid at the surface of the liquid which can be considered to be very small (negligible) if D is large compared to d , so $v_A \approx 0$. v_B is the velocity of the fluid through the orifice at the lower point B. Also, $z_B = 0$ if we measure z from the base of the tank. As mentioned above, the height of the liquid inside the tank will vary with the time t . Hence, we can write $z_A = h(t)$. Using these assumptions, equation (2) can now be written as:

$$P_A + \rho gh(t) = P_B + \frac{1}{2} \rho v_B^2 \dots (3)$$

Since we are interested in the velocity of the liquid at the outlet point B, we re-arrange the above equation and successively, we get:

$$\begin{aligned} \frac{1}{2} \rho v_B^2 &= P_A - P_B + \rho gh(t) \\ v_B^2 &= \frac{2(P_A - P_B)}{\rho} + 2gh(t) \\ v_B &= \left(\frac{2(P_A - P_B)}{\rho} + 2gh(t) \right)^{1/2} \dots (4) \end{aligned}$$

Let $M(t)$ be the mass of the liquid inside the tank at any time t . We know that:

$$\text{Mass} = \text{Volume} \times \text{Density} \dots (5)$$

Hence, the total mass of liquid at any time t can be written as:

$$M(t) = \rho \pi \frac{D^2}{4} h(t) \dots (6)$$

Differentiating this with respect to time t , we get the rate of change of mass inside the tank as:

$$\frac{dM(t)}{dt} = \rho \pi \frac{D^2}{4} \frac{dh(t)}{dt} \dots (7)$$

Now, the principle of the conservation of mass states that:

$$\frac{dM(t)}{dt} = \text{Mass flow into tank} - \text{Mass flow out of tank} \dots (8)$$

Since there is no more liquid entering the tank, we have:

$$\text{Mass flow into the tank} = 0$$

Also, using equation (5) again, we can write:

$$\begin{aligned} \text{Mass flow out of tank} &= \text{Volume flow out of tank} \times \text{density of liquid} \\ &= v_B \times \text{cross-sectional area of orifice} \times \rho \\ &= \rho v_B \pi \frac{d^2}{4} \end{aligned}$$

Substituting the value of v_B from equation (4), we get:

$$\text{Mass flow out of tank} = \rho \pi \frac{d^2}{4} \left(2 \frac{(P_A - P_B)}{\rho} + 2gh(t) \right)^{1/2}$$

So, the conservation equation (8) can now be written as:

$$\frac{dM(t)}{dt} = - \rho \pi \frac{d^2}{4} \left(2 \frac{(P_A - P_B)}{\rho} + 2gh(t) \right)^{1/2} \dots (9)$$

Equating equation (7) and (9), we get:

$$\rho \pi \frac{D^2}{4} \frac{dh(t)}{dt} = - \rho \pi \frac{d^2}{4} \left(2 \frac{(P_A - P_B)}{\rho} + 2gh(t) \right)^{1/2}$$

or

$$\frac{dh(t)}{dt} = - \frac{d^2}{D^2} \left(2 \frac{(P_A - P_B)}{\rho} + 2gh(t) \right)^{1/2} \dots (10)$$

Equation (10) gives the rate of change of the liquid level inside the tank with respect to time. Integrating this equation gives the height of the liquid in the tank at any time t .

EXTENSION ACTIVITY – 1:

By performing a definite integral, show that if the tank pressure and the drain pressure are the same, i.e. $P_A = P_B$, the time taken to empty the cylindrical tank shown in Figure-1 can be represented by the equation:

$$t = \frac{D^2}{d^2} \left(\frac{2h}{g} \right)^{1/2}$$

(Hint: at $t = 0$, the level is h ; and at time t the tank is empty)

EXTENSION ACTIVITY – 2:

Calculate the time taken to empty a tank filled with oil. The tank is 5 m high and has a diameter of 1.5 m. The orifice diameter is 0.1 m. The acceleration due to gravity is 9.81 m/sec². The tank pressure and the orifice outlet pressure are the same.

EXTENSION ACTIVITY – 3:

For the tank in Extension Activity – 2, calculate the time taken for the tank to drain from full to half full.

EXTENSION ACTIVITY – 4:

For the tank in Extension Activity – 2, calculate the orifice diameter required if the full tank must empty in 100 seconds.

WHERE TO FIND MORE

1. *Basic Engineering Mathematics*, John Bird, 2007, published by Elsevier Ltd.
2. *Engineering Mathematics*, Fifth Edition, John Bird, 2007, published by Elsevier Ltd.
3. *Fluid Flow for Chemical and Process Engineers*, 2nd Edition, Holland, F.A & Bragg, R., 1995, published by Butterworth-Heinemann



Tariq Hussain, Senior Advisor- Process Technical Support, MWKL

Tariq is a Chartered Chemical Engineer with 15 years experience in the process design industry, specializing in process dynamic simulation and operator training simulators.

He says:

“Working in the process design industry, we use mathematical models all the time to design and size process equipment such as pumps, compressors, valves and piping. Although most of these models have been wrapped into user-friendly simulation tools, we often have to develop custom mathematical models tailored to deal with unique aspects of each process design project.”

INFORMATION FOR TEACHERS

The teachers should have some knowledge of

- Mathematical modelling with differential equation
- Bernoulli's equation

TOPICS COVERED FROM "MATHEMATICS FOR ENGINEERING"

- Topic 1: Mathematical Models in Engineering
- Topic 2: Models of Growth and Decay
- Topic 6: Differentiation and Integration

LEARNING OUTCOMES

- LO 01: Understand the idea of mathematical modelling
- LO 02: Be familiar with a range of models of change, and growth and decay
- LO 06: Know how to use differentiation and integration
- LO 09: Construct rigorous mathematical arguments and proofs
- LO 10: Comprehend translations of common realistic contexts into mathematics

ASSESSMENT CRITERIA

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 2.2: Set up and solve a differential equation to model a physical situation
- AC 6.1: Calculate the rate of change of a function
- AC 6.2: Use derivatives to classify stationary points of a function
- AC 6.3: Find definite and indefinite integrals of functions
- AC 09.1: Use precise statements, logical deduction and inference
- AC 09.2: Manipulate mathematical expressions
- AC 09.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications.

LINKS TO OTHER UNITS OF THE ADVANCED DIPLOMA IN ENGINEERING

- Unit-1: Investigating Engineering Business and the Environment
- Unit-3: Selection and Application of Engineering Materials
- Unit-5: Maintaining Engineering Plant, Equipment and Systems
- Unit-6: Investigating Modern Manufacturing Techniques used in Engineering
- Unit-7: Innovative Design and Enterprise
- Unit-8: Mathematical Techniques and Applications for Engineers
- Unit-9: Principles and Application of Engineering Science

ANSWERS TO EXTENSION ACTIVITIES

EA1:

Taking equation (10):

$$\frac{dh(t)}{dt} = - \frac{d^2}{D^2} \left(\frac{2(P_A - P_B)}{\rho} + 2gh(t) \right)^{1/2}$$

The inlet and outlet pressure are the same so $(P_A - P_B) = 0$. Therefore:

$$\frac{dh(t)}{dt} = - \frac{d^2}{D^2} (2gh(t))^{1/2}$$

Separating the variables and re-arranging:

$$- dt = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \frac{dh(t)}{(h(t))^{1/2}}$$

Definite Integral:

At time $t = 0$ the height is h , whereas at time t , $h = 0$. So:

$$- \int_t^0 dt = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \int_h^0 \frac{dh(t)}{(h(t))^{1/2}}$$

So, the definite integral becomes:

$$[-t]_0^t = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \left[2(h)^{1/2} \right]_h^0$$

Putting in the integration limits:

$$-t - 0 = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \left[2(0)^{1/2} - 2(h)^{1/2} \right]$$

And rearranging gives:

$$t = \frac{D^2}{d^2} \left(\frac{2h}{g} \right)^{1/2}$$

EA2:

$$t = \frac{1.5^2}{0.1^2} \frac{2}{(2 \times 9.81)^{1/2}} 5^{1/2}$$

$$t = 227.2 \text{ seconds}$$

EA3:

At time $t = 0$ the height is h , whereas at time t , the height is $h/2$. So:

$$- \int_t^0 dt = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \int_{h/2}^h \frac{dh(t)}{(h(t))^{1/2}}$$

which becomes:

$$[-t]_0^t = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \left[2(h)^{1/2} \right]_h^{h/2}$$

Thus:

$$-t - (0) = \frac{D^2}{d^2} \frac{1}{(2g)^{1/2}} \left[2(h/2)^{1/2} - 2(h)^{1/2} \right]$$

and

$$t = \frac{D^2}{d^2} \left(\frac{2h}{g} \right)^{1/2} \left[1 - \left(\frac{1}{2} \right)^{1/2} \right]$$

Substituting in:

$$t = \frac{1.5^2}{0.1^2} \frac{1}{(2 \times 9.81)^{1/2}} \left[2(5)^{1/2} - 2\left(\frac{5}{2}\right)^{1/2} \right]$$

$$t = 66.5 \text{ seconds}$$

EA4:

$$t = \frac{D^2}{d^2} \left(\frac{2h}{g} \right)^{1/2}$$

Rearranging the equation from EA1 in terms of the orifice diameter, d , gives:

$$d^2 = \frac{D^2}{t} \left(\frac{2h}{g} \right)^{1/2}$$

$$d = \sqrt{\frac{D^2}{t} \left(\frac{2h}{g} \right)^{1/2}}$$

Substituting the values:

$$d = \sqrt{\frac{1.5^2}{100} \left(\frac{2 \times 5}{9.81} \right)^{1/2}}$$

$$d = 0.151 \text{ m}$$