

**Please Note**

This exemplar is a bit challenging and an extended version of this is usually introduced in the first year under-graduation engineering courses. But few students may find it interesting. So, give it a go!!

## INTRODUCTION

UK manufacturer JCB's construction machinery can be found on construction sites around the world. JCB is, in fact, the world's third largest producer of construction machinery. JCB's flagship product is the backhoe loader, but their construction equipment products range from micro excavators through to articulated dump trucks.



**Figure-1: Various JCB Construction Vehicles**

Many of JCB's vehicles need to be able to travel across sites and from site to site. Unlike cars, the suspension system for the majority of JCB's machinery is through their large rubber tyres. Additionally, the seat in the cab is sprung to make the ride more comfortable for the operator.

## PROBLEM STATEMENT

It is important to consider the comfort of the driver when designing highway machinery as the operator often has to remain in the same machine for almost 8 hours each day. The vibration amplitudes of the operator's seat are therefore a key factor when designing the cab layout. The vibration amplitudes should be kept as low as possible, so that the operator will have a more satisfying and comfortable experience. Such health and safety issues could be a key factor in selecting heavy construction machinery which requires long operating hours.

## SUSPENSION SYSTEM

A typical installation sees the operator in a machine the total mass of which ( $m_m$ ) is 2500 kg and the tyres act as a spring with effective stiffness ( $k_m$ ) of 500 kN/m. The seat is supported by springs with an effective stiffness ( $k_d$ ) of 10 kN/m. Assume the operator mass ( $m_d$ ) is 100 kg.

During testing and development stage, one of the tests the machines are subjected to is to drive over a surface with a sinusoidal profile. This profile has an amplitude of 20 cm and a wavelength of 2 m. If the machine is driven at an average speed of 18km/hr, we can calculate the vibration amplitude for the driver.

Firstly, a speed of 18 km/hr is converted to 5 m/sec, and we can use the standard wavelength equation to calculate the frequency of the input:

$$v = f\lambda$$

where  $v$  = average speed of the vehicle (m/sec)

$f$  = frequency of testing profile (Hz)

$\lambda$  = wavelength of testing profile (m)

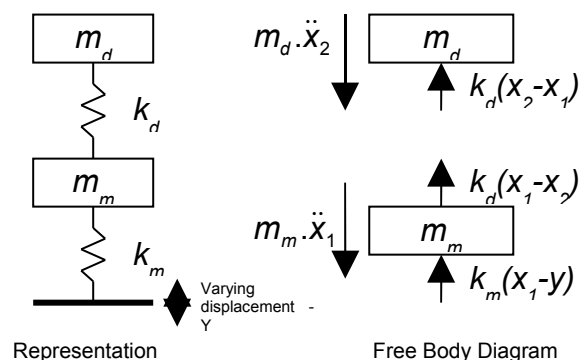
Hence

$$f = \frac{v}{\lambda} = \frac{5}{2} = 2.5 \text{ Hz}$$

In order to proceed, we need to convert the frequency into radians/sec, which is represented by  $\omega$ , where  $\omega = 2\pi f$ . Hence,

$$\omega = 2\pi \times 2.5 = 15.70795 \approx 15.71 \text{ rads/sec}$$

The set up for the system can be represented by the following block diagrams:



Here,  $Y$  represents the amplitude of the testing sinusoidal profile.

## MATHEMATICAL MODELLING

The free body diagram illustrates the forces acting on each mass in accordance with Newton's Law of Motion, which states that:

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

The forces here arise due to the stiffness of the tyres and rubber mounts, and are assumed to follow Hooke's Law, which states that:

$$\text{Force} = \text{Stiffness Coefficient} \times \text{Displacement}$$

For the system described here, it should be assumed that the two masses will move

independently, by different amounts and with different speeds and accelerations. Hence we denote the displacement of the machine mass by 'x<sub>1</sub>' and the displacement of the operator mass by 'x<sub>2</sub>' from their equilibrium positions. Note that x<sub>1</sub> and x<sub>2</sub> are defined to be positive downward.

The forces due to Newton's Law of motion and Hooke's Law of stiffness, described as above, must be in equilibrium. Hence, the equations of motion can be written as:

$$m_m \ddot{x}_1 = -k_m(x_1 - y) - k_d(x_1 - x_2) \dots (1a)$$

$$m_d \ddot{x}_2 = -k_d(x_2 - x_1) \dots (1b)$$

or:

$$m_m \ddot{x}_1 + k_m x_1 + k_d(x_1 - x_2) = k_m y \dots (2a)$$

$$m_d \ddot{x}_2 + k_d(x_2 - x_1) = 0 \dots (2b)$$

These two simultaneous equations can be represented as a matrix equation:

$$\begin{bmatrix} m_m & 0 \\ 0 & m_d \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_m + k_d & -k_d \\ -k_d & k_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_m y \\ 0 \end{bmatrix} \dots (3)$$

To solve this system of equations, we will use the method of substitution. Since the motion is vibrational, displacements  $x$  and  $y$  depend on the angular frequency  $\omega$  and time  $t$ , and an appropriate substitution can be assumed as follows:

$$x = X e^{j\omega t} \dots (4a)$$

$$y = Y e^{j\omega t} \dots (4b)$$

Here  $j$  is the complex number such that  $j = \sqrt{-1}$ . Note that we now use  $X$  and  $Y$  instead of  $x$  and  $y$  to denote the displacement component within the substitution to avoid confusion with the original displacement. Note also that this substitution applies for all displacements.

Differentiating equation (4a) twice with respect to  $t$ , we get:

$$\dot{x} = j\omega X e^{j\omega t} \dots (5a)$$

$$\ddot{x} = -\omega^2 X e^{j\omega t} \dots (5b)$$

Substituting these values from equations (4a), (4b), (5a) and (5b) in equation (3), the matrix equation now becomes:

$$\begin{bmatrix} m_m & 0 \\ 0 & m_d \end{bmatrix} (-\omega^2 e^{j\omega t}) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} k_m + k_d & -k_d \\ -k_d & k_d \end{bmatrix} (e^{j\omega t}) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = e^{j\omega t} \begin{bmatrix} k_m Y \\ 0 \end{bmatrix}$$

Performing matrix addition and cancelling the common term  $e^{j\omega t}$  from both sides, we get:

$$\begin{bmatrix} -\omega^2 m_m + k_m + k_d & -k_d \\ -k_d & -\omega^2 m_d + k_d \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} k_m Y \\ 0 \end{bmatrix}$$

or  $AX = B \dots (6)$

where:

$$A = \begin{bmatrix} -\omega^2 m_m + k_m + k_d & -k_d \\ -k_d & -\omega^2 m_d + k_d \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} k_m Y \\ 0 \end{bmatrix}$$

The solution of equation (6) can be found from:

$$X = A^{-1} B$$

Hence, using the standard form for the inverse of a 2x2 matrix:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{\det \begin{bmatrix} -\omega^2 m_m + k_m + k_d & -k_d \\ -k_d & -\omega^2 m_d + k_d \end{bmatrix}} \times \begin{bmatrix} -\omega^2 m_d + k_d & k_d \\ k_d & -\omega^2 m_m + k_m + k_d \end{bmatrix} \times \begin{bmatrix} k_m Y \\ 0 \end{bmatrix}$$

$$= \frac{1}{(\omega^4 m_m m_d - \omega^2 (m_m k_d + m_d k_m + m_d k_d) + k_m k_d)} \times \begin{bmatrix} k_m Y (-\omega^2 m_d + k_d) \\ k_d k_m Y \end{bmatrix}$$

Comparing both sides, we can find  $X_1$  (the amplitude of the machine displacement) and  $X_2$  (the amplitude of the operator displacement) as follows:

$$X_1 = \frac{k_m Y (-\omega^2 m_d + k_d)}{\omega^4 m_m m_d - \omega^2 (m_m k_d + m_d k_m + m_d k_d) + k_m k_d}$$

$$X_2 = \frac{k_d k_m Y}{\omega^4 m_m m_d - \omega^2 (m_m k_d + m_d k_m + m_d k_d) + k_m k_d}$$

### CONCLUSION

Substituting the values given in this scenario, we can find  $X_2 = 0.6812$  m. Note that, in this instance, it is not important to calculate the magnitude of  $X_1$  since this would represent the vibration characteristics of the machine and not the operator's seat. If it is important to consider this effect, we would normally model the machine vibration as a single mass spring exercise to calculate its vibrational magnitudes.

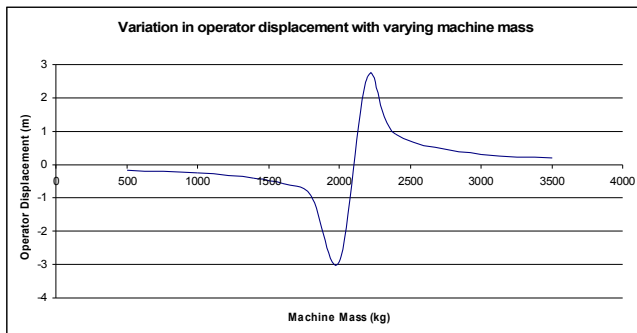
### **EXTENSION ACTIVITY – 1:**

Find the displacement of the operator, correct up to 4 decimal places, when:

1. All variables remain the same but the wavelength of the profile is:
  - (a) 1 m
  - (b) 5 m
  - (c) 10 m
2. All variables remain the same but the amplitude of the profile is:
  - (a) 0.1 m
  - (b) 0.5 m
  - (c) 1 m

### **EXTENSION ACTIVITY – 2:**

Look at the following plot showing how the operator displacement varies when the mass of the machine varies and all other variables remain constant.



- (a) Explain why, for the given set-up, in practice the machine mass should not be chosen to be around 2200 kg even though this gives a displacement of zero.
- (b) Explain why the effective stiffness of the suspension system must be considered in conjunction with the mass of the machine.

### **EXTENSION ACTIVITY – 3:**

Investigate the theory surrounding the use of dampers to moderate the vibration amplitude and explain why the amplitudes calculated in this exercise will alter significantly when system damping is introduced.

N.B. for the purposes of hand calculations, we usually exclude dampers due to the increased number of terms introduced in the calculations. Computer modelling allows us to consider the dampers and hence predict accurately the vibration amplitudes.

### **WHERE TO FIND MORE**

1. *Engineering Mathematics*, Fifth Edition, John Bird, 2007, published by Elsevier Ltd.
2. *Theory and Problems of Mechanical Vibrations: Schaum's Outline Series*, W.W. Setto, 1964, published by Schaum



### **Alan Curtis – Research Engineer**

Alan studied Mechanical Engineering at Loughborough University and graduated in 2006. Since then, Alan has worked for JCB Power Systems working on the design and specification of engines to comply with future emissions legislation. Alan devotes a generous portion of his time to supporting younger engineers at various stages of their education.

### INFORMATION FOR TEACHERS

The teachers should have some knowledge of

- Newton's laws of motion and Hooke's law of elasticity
- How to solve simultaneous equations using the matrix method followed by the substitution method that involves first and second derivatives
- How to expand a determinant and find the inverse of a matrix

### TOPICS COVERED FROM "MATHEMATICS FOR ENGINEERING"

- Topic 1: Mathematical Models in Engineering
- Topic 4: Functions
- Topic 6: Differentiation and Integration
- Topic 7: Linear Algebra and Algebraic Processes

### LEARNING OUTCOMES

- LO 01: Understand the idea of mathematical modelling
- LO 04: Understand the mathematical structure of a range of functions and be familiar with their graphs
- LO 06: Know how to use differentiation and integration in the context of engineering analysis and problem solving
- LO 07: Understand the methods of linear algebra and know how to use algebraic processes
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

### ASSESSMENT CRITERIA

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 4.1: Identify and describe functions and their graphs
- AC 6.1: Calculate the rate of change of a function
- AC 7.2: Use matrices to solve two simultaneous equations in two unknowns
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

### LINKS TO OTHER UNITS OF THE ADVANCED DIPLOMA IN ENGINEERING

- Unit-1: Investigating Engineering Business and the Environment
- Unit-3: Selection and Application of Engineering Materials
- Unit-4: Instrumentation and Control Engineering
- Unit-5: Maintaining Engineering Plant, Equipment and Systems
- Unit-6: Investigating Modern Manufacturing Techniques used in Engineering
- Unit-7: Innovative Design and Enterprise
- Unit-8: Mathematical Techniques and Applications for Engineers
- Unit-9: Principles and Application of Engineering Science

### ANSWERS TO EXTENSION ACTIVITIES

**EA1:** 1(a) 0.0058 m; 1(b) 0.4185 m; 1(c) 0.2340 m; 2(a) 0.3406 m; 2(b) 1.7031 m; 2(c) 3.4062 m

### **EA2:**

- (a) Ideally we would design the machine to weigh 2200 kg where the vibration amplitude of the driver would be 0. However, if the machine mass alters slightly from this then the vibration amplitudes would increase significantly to maxima at 2000 kg and 2400 kg. Hence it would make more sense to design the machine mass to be around 1500 kg or 3000 kg where a slight variation would not dramatically affect the vibration amplitude of the operator.
- (b) In most instances the mass of the machine can only be varied by small amounts within the design. With a given suspension system and effective stiffness this gives very little control over the displacement of the operator. If the two are considered together then the suspension system can be refined to eliminate excessive oscillation of the operator.

**EA3:** Damping will restrict the vibration magnitude and hence the large displacements will more than likely not be seen. Even if the peak vibration amplitudes do occur, the dampers will mean that the motion is much smoother and therefore not perceived as negatively by the operator.