

INTRODUCTION

BLOODHOUND SSC is an engineering education adventure set up not only to achieve an iconic land speed record but to stimulate young people to consider engineering as a career choice. At such high speeds the wheels of BLOODHOUND SSC are subjected to enormous stresses. Designing and constructing the wheels is a huge engineering challenge as it is not possible to use existing technologies. Aluminium wheels will not be strong enough but nobody has ever cast such a big disc of titanium which is the material of choice. The wheels must be light to reduce inertia, large to reduce the angular velocity and strong to withstand the stresses.

IMPORTANCE OF EXEMPLAR IN REAL LIFE

Land transport relies on wheels. This ancient invention continues to be the bedrock of most transport systems. Engines and gear-boxes are typically full of rotating machinery and understanding the loads imposed on components is of great importance for safety considerations.

SCENARIO

We will derive the acceleration of a point on the rim of a wheel for an observer moving with the wheel. Knowing the acceleration, it is possible to calculate forces and stresses within the wheel. With an extension activity at the end, we will see that the knowledge of frames of reference is very useful for the engineer as problems can frequently be made simpler by choosing the appropriate reference frame.

MATHEMATICAL MODEL

Consider a point on the rim of a rotating wheel whose centre is stationary in the frame of reference of the observer who is also moving with the wheel. Consider the plane of rotation to be vertical. We can clearly visualise that the locus of such point is a circle and is shown in Figure 1 below:

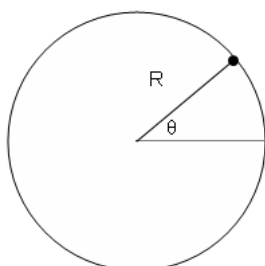


Figure 1: Locus of a point on the rim of the wheel for an observer moving with the wheel

The parametric equations of this point on the circle are given by:

$$x = R \cos \theta \text{ and } y = R \sin \theta \dots (1)$$

where R is the radius of this circle and θ is the angle as shown in Figure 1. Here, x describes the position of the moving point in horizontal direction whereas y describes its position in the vertical direction. Using Pythagoras' theorem, the displacement of the point from the centre of the circle, which is nothing but the radius of the circle, is given by:

$$R = \sqrt{x^2 + y^2} \dots (2)$$

If the angular velocity of this moving point is ω , then:

$$\theta = \omega t \dots (3)$$

at any time $t > 0$.

Substituting in equation (1), we get

$$x = R \cos \omega t \text{ and } y = R \sin \omega t \dots (4)$$

Differentiating equation (4) twice with respect to time t , we get subsequently:

$$\dot{x} = -\omega R \sin \omega t \text{ and } \dot{y} = \omega R \cos \omega t \dots (5)$$

$$\ddot{x} = -\omega^2 R \cos \omega t \text{ and } \ddot{y} = -\omega^2 R \sin \omega t \dots (6)$$

Equation (5) and (6) represents the velocity and acceleration, respectively, of this moving point on the circle in the horizontal and vertical directions. Using Pythagoras' theorem, we can find the magnitude of the resultant acceleration of this point as follows:

$$\begin{aligned} \text{Acceleration} &= \sqrt{(\ddot{x})^2 + (\ddot{y})^2} \\ &= \sqrt{(-\omega^2 R \cos \omega t)^2 + (-\omega^2 R \sin \omega t)^2} \\ &= \sqrt{\omega^4 R^2 (\cos^2 \omega t + \sin^2 \omega t)} \\ &= \omega^2 R \end{aligned} \dots (7)$$

If v is the tangential velocity of this point moving in a circular path with radius R and the angular velocity is ω , then we know the following relationship:

$$v = R\omega \text{ or } \omega = \frac{v}{R}$$

Using this in equation (7), we can write:

$$\text{Acceleration} = \frac{v^2}{R} \dots (8)$$

Newton succeeded in showing geometrically that an object performing circular motion would experience an acceleration of magnitude (v^2/R) directed towards the centre of the circle known as the centripetal acceleration. We have successfully shown the same relationship here.

EXTENSION ACTIVITY – 1:

Calculate the acceleration of a point on the rim of BLOODHOUND SSC's wheels at full speed. Given that the diameter of the wheel is 0.9m and full speed is 1000mph (don't forget to convert it in m/sec.)

If a 1 kg bag of sugar was spun at this rate what would its effective weight be?

Why this knowledge is important for the designers of BLOODHOUND's wheels? Hence why is Titanium being used and not Aluminium?

EXTENSION ACTIVITY – 2:

Consider the wheel rotating along the ground and the same fixed point on its rim is now being observed by an observer who is stationary on the ground. The locus of this point to this stationary observer is not a circle any more but a *cycloid*.



Figure 2: Locus of a point on the rim of the wheel for an observer on the ground

Both the observers are in different frames of reference which are in uniform relative motion. The relative velocity of frames of reference with respect to each other is v .

Show that the parametric equations of the locus of the point executing a cycloid are given by

$$x = Vt - R \sin \omega t \text{ and } y = R - R \cos \omega t$$

where V is the horizontal speed of the wheel, t is the time elapsed and $\theta = \omega t$ is the angle between the radius from the point and the vertical axis. This is shown in the following figure 3.

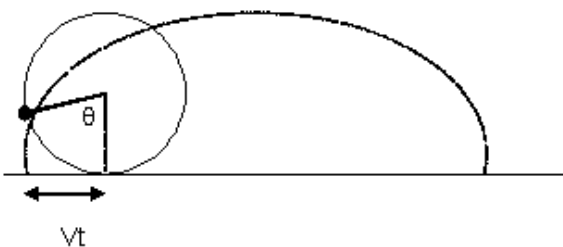


Figure 3: Position of the wheel after time t moving with velocity V.

EXTENSION ACTIVITY – 3:

Since the laws of Physics are the same for both observers, further to the Extension Activity 2, show that the acceleration of the point on the rim of the wheel should be the same for both observers, i.e. the acceleration in this case will be same as

$$\text{Acceleration} = \omega^2 R$$

EXTENSION ACTIVITY – 4:

For cycloid motion, observe that the point executing cycloids is momentarily at rest each time it touches the ground. With this knowledge, find an expression for the magnitude of the velocity $|u|$, where u is the velocity of the point on the rim.

Hence show that:

- $|u| = 0$, each time the point is in contact with the ground, and
- $|u| = 2V$ each time the point is at "the" top of the cycloid.

(Hint: at the ground $\theta = 0, 2\pi, 4\pi, \dots$ etc.)

CONCLUSION

In this exemplar, we have shown that the acceleration of an object is the same from two different inertial frames of reference. This is important as otherwise the laws of physics would be different in the two different frames. The same force would result in two different accelerations! However, the velocities are different in the two different frames, motion is simply relative!

An inertial frame of reference is one in which Newton's Laws work. These laws also work in any other frame of reference which is in uniform relative motion with the original. Hence all these other frames are also inertial frames.

WHERE TO FIND MORE

- Basic Engineering Mathematics*, John Bird, 2007, published by Elsevier Ltd.
- Engineering Mathematics*, Fifth Edition, John Bird, 2007, published by Elsevier Ltd.
- <http://www.bloodhoundssc.com>



**Ian Galloway – Deputy Director, Science Learning Centre South East
Education Director, Bloodhound SSC**

He says:

“The language of nature is mathematics. Things have not changed since the time of Galileo and being able to make predictions is based on our ability to construct mathematical models of the world around us”.

INFORMATION FOR TEACHERS

The teachers should have some knowledge of

- Frame of references
- Parametric equations of circle and cycloid
- Velocity and acceleration of a moving object in a circular and cycloidal path

TOPICS COVERED FROM “MATHEMATICS FOR ENGINEERING”

- Topic 1: Mathematical Models in Engineering
- Topic 3: Models of Oscillations
- Topic 4: Functions
- Topic 5: Geometry
- Topic 6: Differentiation and Integration
- Topic 7: Linear Algebra and Algebraic Processes

LEARNING OUTCOMES

- LO 01: Understand the idea of mathematical modelling
- LO 03: Understand the use of trigonometry to model situations involving oscillations
- LO 04: Understand the mathematical structure of a range of functions and be familiar with their graphs
- LO 05: Know how 2-D and 3-D coordinate geometry is used to describe lines, planes and conic sections within engineering design and analysis
- LO 06: Know how to use differentiation and integration in the context of engineering analysis and problem solving
- LO 07: Understand the methods of linear algebra and know how to use algebraic processes
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

ASSESSMENT CRITERIA

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 3.1: Solve problems in engineering requiring knowledge of trigonometric functions
- AC 3.2: Relate trigonometric expressions to situations involving oscillations
- AC 4.1: Identify and describe functions and their graphs
- AC 5.1: Use equations of straight lines, circles, conic sections, and planes
- AC 6.1: Calculate the rate of change of a function
- AC 7.1: Solve engineering problems using vector methods
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

LINKS TO OTHER UNITS OF THE ADVANCED DIPLOMA IN ENGINEERING

- Unit-1: Investigating Engineering Business and the Environment
- Unit-3: Selection and Application of Engineering Materials
- Unit-4: Instrumentation and Control Engineering
- Unit-5: Maintaining Engineering Plant, Equipment and Systems
- Unit-6: Investigating Modern Manufacturing Techniques used in Engineering
- Unit-7: Innovative Design and Enterprise
- Unit-8: Mathematical Techniques and Applications for Engineers
- Unit-9: Principles and Application of Engineering Science

ANSWERS TO EXTENSION ACTIVITIES

Please contact Dr Sapna Somani at sapna.somani@raeng.org.uk for the answers in detail.